In this lecture we prove the law of iterated logarithm; First we prove the following lemma:

(i) (LCLT - local CLT) If \( k = o \left( n^{3/4} \right) \) and \( n + k \) is even, then \( P \left( S_n = k \right) \downarrow \frac{\sqrt{2}}{\pi n} e^{-\frac{k^2}{2n}} \).

(ii) For all \( n, k \geq 0 \), \( P \left( S_n \geq k \right) \leq e^{-\frac{k^2}{2n}} \) (not tight, by a polynomial factor).

Then by using the Borel-Cantelli lemma we show that \( \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \) a.s.,
and by symmetry, \( \liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1 \) a.s.

We discuss Higher dimensional RW and general 1D RW;

We prove the Hewitt-Savage 0-1 law: If \( A \in \epsilon \) (for a RW) then \( P \left( A \right) \in \{ 0, 1 \} \),
and its following application: For a RW in \( \mathbb{R} \), exactly one of the following has probability 1:

(i) \( S_n = 0 \) for all \( n \) (trivial RW)
(ii) \( S_n \to \infty \)
(iii) \( S_n \to -\infty \)
(iv) \( \limsup S_n = \infty \) and \( \liminf S_n = -\infty \)

1 Law of the iterated logarithm

For SRW, we know by CLT that \( S_n \approx N \left( 0, n \right) \).

Is it true that \( P \left( S_n = k \right) \approx \frac{1}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}} \) (density at \( k \) of \( N \left( 0, n \right) \))? For parity reasons, this is false; \( P \left( S_n = k \right) = 0 \) if \( k + n \) is odd.

Is it true that \( P \left( S_n = k \right) \approx \frac{2}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}} \) when \( k + n \) is even (and \( n \) is large)?
Yes, this is true.
Lemma.  
(i) (LCLT - local CLT) If $k = o\left(n^{3/4}\right)$ and $n + k$ is even, then

$$P\left(S_n = k\right) \sim \frac{\sqrt{2\pi n} e^{-\frac{k^2}{2n}}}{\sqrt{2\pi n} \sqrt{2\pi n-k} \frac{n+k}{2} \frac{n-k}{2}}.$$  

RemarK: This estimate is uniform for $k = o\left(n^{3/4}\right)$.

(ii) For all $n, k \geq 0$, $P\left(S_n \geq k\right) \leq e^{-\frac{k^2}{2n}}$ (not tight, by a polynomial factor).

Proof.  
(i)

$$P\left(S_n = k\right) = \left(\frac{n}{2}\right)^{2-n} \left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!$$

Stirling

$$\sim \frac{\sqrt{2\pi n} n^n}{\sqrt{2\pi n} \sqrt{2\pi n-k} \frac{n+k}{2} \frac{n-k}{2}}$$

$$= \left[\frac{2}{\pi n \left(1 - \frac{k^2}{n^2}\right)} \cdot \frac{n^n}{(n+k) \frac{n+k}{2} (n-k) \frac{n-k}{2}}\right]$$

$$= \exp\left(n \log n - \frac{n+k}{2} \log (n+k) - \frac{n-k}{2} \log (n-k)\right)$$

$$n \log n - \frac{n+k}{2} \log (n+k) - \frac{n-k}{2} \log (n-k) \triangleq (*)$$

Since $\log (n+k) = \log (n) + \log (1 + \frac{k}{n})$,

and since $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - O\left(x^4\right)$,

$$\log (1 + x) \xrightarrow[x \to 0]{} \frac{x}{1}$$

$$(*) = -\frac{n+k}{2} \log \left(1 + \frac{k}{n}\right) - \frac{n-k}{2} \log \left(1 - \frac{k}{n}\right) =$$

$$= -\frac{n+k}{2} \left(\frac{k}{n} - \frac{k^2}{2n^2} + \frac{k^3}{3n^3} + O\left(\frac{k^4}{n^4}\right)\right) - \frac{n-k}{2} \left(\frac{k}{n} + \frac{k^2}{2n^2} + \frac{k^3}{3n^3} + O\left(\frac{k^4}{n^4}\right)\right) =$$

$$= \frac{k^2}{4n} + \frac{k^2}{4n} - \frac{k^2}{2n} + O\left(\frac{k^4}{n^3}\right) = \frac{k^2}{2n} + O\left(\frac{k^4}{n^3}\right) \xrightarrow[k = o\left(n^{3/4}\right)]{n \to \infty}$$

Plugging back into $(*)$: 3-2
\[ P(S_n = k) \sim \sqrt{\frac{2}{\pi n(1 - \frac{k^2}{n^2})}} e^{-\frac{k^2}{2n} + O\left(\frac{k^4}{n^4}\right)} \sim \sqrt{\frac{2}{\pi n}} e^{-\frac{k^2}{2n}} \quad \left( n \to \infty \right) \]

(ii) We will use that for any \( X \),

\[ P(X \geq t) = \frac{\theta > 0}{\text{Markov}} \frac{E e^{\theta X}}{e^{\theta t}} \]

Taking \( X = S_n \), \( t = k \) and by comparing Taylor coefficients,

\[ E e^{\theta X_1} = \left( \frac{1}{2} e^\theta + \frac{1}{2} e^{-\theta} \right) \leq e^{\theta^2/2} \]

\[ P(S_n \geq k) \leq \frac{E e^{\theta S_n}}{e^{\theta k}} = \left( \frac{1}{2} e^\theta + \frac{1}{2} e^{-\theta} \right)^n \leq \frac{e^{\theta^2 n/2}}{e^{\theta k}} \]

\[ \text{take } \theta = \frac{k}{n} \]

\[ \square \]

Lemma. For \( k > \sqrt{n} \), \( k = o\left(\frac{n^{3/4}}{}\right) \) we have \( P(S_n \geq k) \geq c e^{-k^2/2n} \) for some \( c > 0 \).

Remark: If \( |k| \leq \sqrt{n} \), then \( P(S_n \geq k) \geq c \) for some \( c > 0 \) by the CLT.

Proof.

\[ P(S_n \geq k) \geq \frac{E e^{\theta S_n}}{e^{\theta k}} = \frac{\left( \frac{1}{2} e^\theta + \frac{1}{2} e^{-\theta} \right)^n}{e^{\theta k}} \]

For these \( j \) we have

\[ e^{-j^2/2n} \geq e^{-\left(k + \frac{n}{2}\right)^2/2n} \geq e^{-\frac{k^2}{2n} - 1 - \frac{n}{2k} \geq e^\theta e^{-k^2/2n} \]

Thus,

\[ P(S_n \geq k) \geq \frac{c''}{\sqrt{n}} e^{-k^2/2n} \cdot \frac{n}{k} \]

\[ \square \]

Reminder: Define \( M_n = \max_{0 \leq j \leq n} S_j \). Then \( P(M_n \geq k) = P(S_n = k) + 2P(S_n > k) \).

\( S_n \) is typically of order \( \sqrt{n} \). By (ii) and Borel-Cantelli, a.s. \( S_n \leq \sqrt{2n \log n} (1 + \epsilon) \) eventually.
Lemma. (Borel-Cantelli)
If \{A_n\} is a sequence of events, then:
1) If \(\sum P(A_n) < \infty\), then a.s. only finitely many occur.
2) If \(\sum P(A_n) = \infty\) and \(A_n\) are independent, then a.s. infinitely many of them occur.

Application: Define \(A_n = \{S_n \geq (1 + \epsilon) \sqrt{2n \log n}\}\). By (ii), \(P(A_n) \leq \frac{1}{n^{1+\epsilon}}\).
So \(\sum P(A_n) < \infty\), and therefore a.s. \(S_n \leq (1 + \epsilon) \sqrt{2n \log n}\) eventually.

Theorem. (LIL)
\[\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1\] a.s.

Remark: By symmetry, \(\liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1\) a.s.

Notation: \(u(n) = \sqrt{2n \log \log n}\)

Proof. Remark: For ease of readability, we will not worry about integrality of indices in the proof. One may round the indices appropriately everywhere and the argument will still go through.

For \(\gamma > 0, a > 1, k \in \mathbb{N}\)

\[P\left(\max_{0 \leq j \leq a^k} S_j \geq (1 + \gamma) u(a^k)\right) \leq 2P\left(S_{a^k} \geq (1 + \gamma) u(a^k)\right) \leq 2e^{-\frac{(1+\gamma)^2u(a^k)^2}{2a^k}} = 2e^{-\frac{(1+\gamma)^2 \log \log (a^k)}} = \frac{2}{(k \log a)^{(1+\gamma)^2}}\]

These probability estimates are summable in \(k\). Thus by B-C,

\[P\left(\max_{0 \leq j \leq a^k} S_j \leq (1 + \gamma) u(a^k) \text{ from some } k \text{ on}\right) = 1\]

Now, for large \(n\), write \(a^{k-1} \leq n \leq a^k\). Then

\[\frac{S_n}{u(n)} = \frac{S_n}{u(a^k)} \cdot \frac{u(a^k)}{a^k} \cdot \frac{a^k}{n} \cdot \frac{n}{u(n)} \leq a(1 + \gamma)\]
since \( \frac{S_n}{u(a^k)} \leq 1 + \gamma \cdot \frac{a^k}{n} \leq a \), and \( \frac{u(a^k)}{a^k} \cdot \frac{n}{u(n)} \leq 1 \) because \( \frac{u(t)}{t} \) is eventually decreasing.

We deduce that

\[
P\left( \max_{0 \leq j \leq a^k} S_j \leq (1 + \gamma) u(a^k) \text{ from some } k \text{ on} \right) = 1
\]

\[
P\left( \limsup_{n} \frac{S_n}{u(n)} \leq a(1 + \gamma) \right) = 1
\]

Since \( \gamma > 0 \) and \( a > 1 \) are arbitrary, we have

\[
P\left( \limsup_{n} \frac{S_n}{u(n)} \leq 1 \right)
\]

For the lower bound, fix \( 0 < \gamma < 1, a > 1 \).

Denote \( A_k = \{ S_{a^k} - S_{a^{k-1}} \geq (1 - \gamma) u(a^k - a^{k-1}) \} \).

We need to show that \( \sum P(A_k) = \infty \).

Denote: \( n = a^k - a^{k-1} \).

For large \( k \),

\[
P(A_k) \geq \frac{c}{(1 - \gamma) u(n)} \cdot \frac{1}{\sqrt{2} \log \log n} \cdot \frac{1}{(\log n)^{(1 - \gamma)^2}}
\]

Noting \( \log n \approx k \). So this expression is not summable in \( k \).

Therefore, by B-C,

\[
P(\text{infinitely many of } A_k \text{ occur}) = 1
\]

By the upper bound,

\[
P\left( \liminf_{n} \frac{S_n}{u(n)} \geq -1 \right) = 1
\]

So we have, for large \( k \),
Thus

\[ P\left(\limsup_{n} \frac{S_n}{u(n)} \geq (1 - \gamma) \sqrt{1 - \frac{1}{a}} - \frac{1 + \epsilon}{\sqrt{a}}\right) = 1 \]

and we take \( \gamma \to 0 \) and \( a \to \infty \) to obtain the lower bound. \( \Box \)

## 2 Higher dimensional RW and general 1D RW

In 1D and in 2D SRW is recurrent. In 3D (and more D) it is transient.

For a general RW in \( \mathbb{R}^d \) (\( S_n = X_1 + \cdots + X_n, \text{ I.I.D.}, X_1 \in \mathbb{R}^d \)), what can we say about \( P(S_n = 0 \text{ infinitely often}) \)? It turns out that this probability is 0 or 1.

Say that an event \( A \) is \textit{permutable} if the occurrence of \( A \) is unaffected by applying a \( \text{finite permutation} \) (a function \( \pi : \mathbb{N} \to \mathbb{N} \) s.t. \( \pi(n) = n \) for all \( n \geq n_0 \)) to the increments of \( S_n \).

More formally, if the increments take values in a state space \( S \), let our probability space \( \Omega \) be \( S^{\mathbb{N}} \), with the product probability over increments.

An event \( A \in \mathcal{F} (\sigma\text{-field}) \) is \textit{permutable} if \( A = \{ \omega \in \Omega | \pi(\omega) \in A \} \) for all finite permutations \( \pi \).

The collection of all permutable events forms \( \sigma\text{-field} \) \( \varepsilon \), the \textit{exchangable} \( \sigma\text{-field} \).

**Theorem.** \( (\text{Hewitt-Savage 0-1 law}) \)

If \( A \in \varepsilon \) (for a RW) then \( P(A) \in \{0, 1\} \).

More Examples:

1. For any \( B \), \( \{S_n \in B \text{ infinitely often} \} \in \varepsilon \).
2. For any $C_n$, \( \limsup_{n \to \infty} \frac{S_n}{C_n} \geq 1 \) \in \varepsilon.

3. Any tail event is permutable (tail $\sigma$-field $\subseteq \varepsilon$).

A tail event is an event which, for any $n$, is a function only of $X_n, X_{n+1}, \cdots$.

**Proof.** Fix $A \in \varepsilon$. We will show that $P(A) = P(A)^2$ (in other words, $A$, is independent of itself).

Take a sequence of events $A_n$ s.t. $A_n$ is a function only of $X_1, \cdots, X_n$ and:

(1) \[ P(A_n \Delta A) \to 0 \]

(Xor: $B \triangle C = (B \cup C) \setminus (B \cap C)$)

Let $\pi = \pi_n$ be the permutation which exchanges $1, \cdots, n$ with $n + 1, \cdots, 2n$.

Write $A'_n = \pi(A_n)$. Notice that $A_n$ and $A'_n$ are independent.

Since $\pi$ preserves probability (since $X_1, \cdots$ are I.I.D.),

(2) \[ P(A_n \Delta A) = P(\pi(A_n \Delta A)) \stackrel{A \text{ is permutable}}{=} P(A'_n \Delta A) \]

Noticing that $|P(B) - P(C)| \leq P(B \triangle C)$, then (1) and (2) imply that $P(A_n) \to P(A)$ and

(3) \[ P(A'_n) \to P(A) \]

However, we also obtain that

\[ P(A_n \Delta A'_n) \leq P(A_n \Delta A) + P(A'_n \Delta A) \to 0 \]

from which

\[ 0 \leq P(A_n) - P(A_n \cap A'_n) \leq P(A_n \cup A'_n) - P(A_n \cap A'_n) = P(A_n \Delta A'_n) \to 0 \]

But $P(A_n) \to P(A)$, and by independence $P(A_n) P(A'_n) \to P(A)^2$. Thus,
\[ P(A) = P(A)^2 \]

as we wanted. \(\square\)

**Application:** For a RW in \(\mathbb{R}\), exactly one of the following has probability 1:

(i) \(S_n = 0\) for all \(n\) (trivial RW)

(ii) \(S_n \to \infty\)

(iii) \(S_n \to -\infty\)

(iv) \(\limsup S_n = \infty\) and \(\liminf S_n = -\infty\)

**Proof.** By the 0-1 law, \(\limsup_{n \to \infty} S_n\) is a constant \(c \in [-\infty, \infty]\) (since it is a permutable RV).

Notice that by considering the first increment \(X_1\), \(c = c - X_1\).

Thus, either \(X_1 \equiv 0\) (this is case (i)), or \(c \in \{-\infty, \infty\}\).

Similarly, \(\liminf S_n \in \{-\infty, \infty\}\). \(\square\)

**Exercise:** If \(X_1 \in \mathbb{R}\) is non-degenerate (not always 0: \(P(X_1 = 0) < 1\)), then we are in case (iv) if either:

1. \(X_1\) is symmetric.

2. \(E(X_1) = 0\) and \(E(X_1^2) < \infty\). (It is actually true that \(E(X_1) = 0\) suffices, as we will see next time.)