Random Walks and Brownian Motion Tel Aviv University Spring 2011	Instructor: Ron Peled

Lecture 4

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This lecture deals primarily with recurrence for general random walks. We present several criteria for a random walk to be recurrent, and prove Polya's theorem on recurrence and transience for the simple random walk on $\mathbb{Z}^d$

1 Recurrence for general random walks

Remember the following result from the previous lecture: Every 1-dimensional random walk S_n satisfies exactly one of the following almost surely:

(i) $S_n = 0$ for every n.

(ii) $\lim S_n = \infty$.

(iii) $\lim S_n = -\infty$.

(iv) $\limsup S_n = \infty$ and $\liminf S_n = -\infty$.

We would like to contrast this with the following definition:

Definition 1 *A random walk taking values in* $\mathbb{R}^d$ *is called* point-recurrent *if*

$$\mathbf{P}(S_n = 0 \text{ infinitely often}) = 1.$$

(remember this probability is either 0 *or* 1*, by the Hewitt-Savage zero-one law).*

We also define the set of possible values *for the random walk as the set of all* $x \in \mathbb{R}^d$ *such that* $\mathbf{P}(S_n = x) > 0$ *for some* n*.*

Exercise If a random walk is point-recurrent, then

$$\mathbf{P}(S_n = x \text{ infinitely often}) = 1$$

for every possible value x.

Remark 2 *If the random walk is not discrete (having an atomic distribution for each step) then these definitions are not very useful. Instead we say that the random walk is* neighborhood-recurrent *if for some (and then for any, as can be proved)* $\epsilon > 0$

$$\mathbf{P}(|S_n| < \epsilon \text{ infinitely often}) = 1.$$

Similary, possible values are changed to those x for which for every $\epsilon > 0$ there exists an n such that $\mathbf{P}(|S_n - x| < \epsilon) > 0$.

Proposition 3 *A random walk in $\mathbb{R}^d$ is point-reucrrent if and only if one of the following holds:*

(i) $\mathbf{P}(\exists n \geq 1,\ S_n = 0) = 1$.

(ii) $\mathbf{P}(S_n = 0 \text{ infinitely often}) = 1$ *(this is the definition of point-recurrence).*

(iii) $\sum_{n=0}^{\infty} \mathbf{P}(S_n = 0) = \infty$.

Proof The equivalence between (i) and (ii) is clear. For the equivalence between (i) and (iii), note that by the strong markov property, the number of visits to 0 is distributed geometrically $\text{Geo}(p)$, where p is the probability of no return to 0. Therefore

$$\sum_{n=0}^{\infty} \mathbf{P}(S_n = 0) = \mathbf{E}\left(\text{Geo}(p)\right) = \frac{1}{p},$$

and now the equivalence is obvious. $\square$

Theorem 4 (Pólya) *A simple random walk in $\mathbb{Z}^d$ is recurrent if and only if $d = 1$ or $d = 2$.*

Proof We consider each dimension separately:

$\mathbf{d = 1}$: We already saw this result in dimension 1, but we now give a different proof. Notice that by Stirling's formula

$$\mathbf{P}(S_{2n} = 0) = 2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

Therefore $\sum_{n=0}^{\infty} \mathbf{P}(S_{2n} = 0) = \infty$ and we are done.

$\mathbf{d = 2}$: By rotating $\mathbb{Z}^2$ in 45° we get that each step is like moving one step in each of two independent, one dimensional, simple random walks. Hence

$$\mathbf{P}(S_{2n} = 0) \sim \left(\frac{1}{\sqrt{\pi n}}\right)^2 = \frac{1}{\pi n},$$

and again $\sum_{n=0}^{\infty} \mathbf{P}(S_{2n} = 0) = \infty$.

d $= \mathbf{3}$: We calculate explicitly:

$$\mathbf{P}(S_{2n}=0) = 6^{-2n} \sum_{\substack{j,k\geq 0\\ j+k\leq n}} \frac{(2n)!}{(j!\cdot k!\cdot (n-j-k)!)^2}$$
$$= 2^{-2n}\binom{2n}{n} \sum_{\substack{j,k\geq 0\\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot (n-j-k)!}\right)^2$$

(here j is the number of steps in direction $(1,0,0)$, and k is the number of steps in direction $(0,1,0)$). Since

$$\sum_{\substack{j,k\geq 0\\ j+k\leq n}} 3^{-n}\frac{n!}{j!\cdot k!\cdot (n-j-k)!} = 1,$$

it follows that

$$\mathbf{P}(S_{2n}=0) \leq 2^{-2n}\binom{2n}{n} \max_{\substack{j,k\geq 0\\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot (n-j-k)!}\right).$$

The maximum is achieved when j and k are as close as possible to $\frac{n}{3}$, and this maximum is smaller than $\frac{c}{n}$ for some constant c. Hence

$$P(S_{2n}=0) \leq \frac{c}{n^{\frac{3}{2}}},$$

so $\sum_{n=0}^{\infty} \mathbf{P}(S_{2n}=0) < \infty$ and the random walk is transient.

d $\geq \mathbf{4}$: The random walk is still transient, since the first 3 coordinates are transient. $\square$

Exercise Find a point-recurrent random walk on $\mathbb{R}$ whose set of possible values is a countable dense set in $\mathbb{R}$.

2 The Chung-Fuchs theorem

Theorem 5 (Chung-Fuchs, 1951) *Let S_n be a random walk in $\mathbb{R}^d$. Then:*

(i) If $d=1$ and $\frac{S_n}{n}\to 0$ in probability, then S_n is neighborhood-recurrent. In particular, this happens if $\mathbf{E}X_1 = 0$.

(ii) *If* $d = 2$ *and* $\frac{S_n}{\sqrt{n}}$ *converges in distribution to a centered normal distribution, then* S_n *is neighborhood-recurrent. In particular, this happens if* $\mathbf{E}X_1 = 0$ *and* $\mathbf{E}X_1^2 < \infty$.

(iii) *If* $d = 3$ *and the random walk is not contained in a plane then it is neighborhood-transient (the condition just means that the set of neighborhood-possible values is not contained in a plane).*

Remark 6 *If the walk is on* $\mathbb{Z}^d$ *then neighborhood-recurrence is the same as point-recurrence and the theorem still applies.*

Remark 7 *If* $\mathbf{E}X_1 = \mu$ *then by the strong law of large numbers* $\frac{S_n}{n} \to \mu$ *almost surely. Therefore if* $\mathbf{E}X_1$ *exists and is nonzero, it's obvious that the random walk is transient.*

We won't prove the Chung-Fuchs theorem in its full generality, but we will show some partial results, together with other criteria for recurrence. We will need the following theorem:

Theorem 8 (Birkhoff ergodic theorem for functions of IIDs) *Let* $X_1, X_2, \ldots$ *be IID random variables in a state space* S *and let* $g : S^\infty \to \mathbb{R}$ *be any measureable function. Define*

$$Y_n = g(X_n, X_{n+1}, \ldots).$$

If $\mathbf{E}|Y_1| < \infty$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = \mathbf{E}Y_1$$

almost surely and in L^1.

Remark 9 *In fact, it is sufficient to take the* Y_n*'s to be a stationary ergodic sequence with* $\mathbf{E}|Y_1| < \infty$

We won't prove this theorem (a proof can be found, for example, in [1]). However, we will use this result to prove the following:

Theorem 10 (Kesten-Spitzer-Whitman range theorem, 1964) *For a random walk in* $\mathbb{R}^d$ *let*

$$R_n = |\{x :\ \exists k \le n, S_k = x\}|$$

be the number of distinct points visited, then

$$\mathbf{P}(\textit{no return to 0}) = \lim_{n\to\infty} \frac{R_n}{n}$$

almost surely.

Remark 11 *This theorem also holds for ergodic markov chains and even stationary ergodic sequences.*

Proof Note that

$$R_n = \sum_{k=0}^{n} \mathbf{1}[S_j \neq S_k \text{ for } k+1 \leq j \leq n].$$

Thus

$$R_n \geq \sum_{k=0}^{n} \mathbf{1}[S_k \text{ never revisited}] = \sum_{k=0}^{n} g(X_{k+1}, X_{k+2}, \ldots)$$

for an appropriate measureable function g. Now we can use Birkhoff ergodic theorem and get that

$$\liminf_{n\to\infty} \frac{R_n}{n} \geq \mathbf{E}g(X_1, X_2, \ldots) = \mathbf{P}(\text{no return to } 0)$$

almost surely.

On the other hand, fix $M \geq 1$ and note that

$$R_n \leq \sum_{k=0}^{n-M} \mathbf{1}\left[S_k \text{ not visited again by time } k+M\right] + M.$$

Again by Birkhoff we get that

$$\limsup_{n\to\infty} \frac{R_n}{n} \leq \limsup_{n\to\infty} \frac{1}{n}\left(M + \sum_{k=0}^{n-M} \mathbf{1}\left[S_k \text{ not visited again by time } k+M\right]\right) =$$
$$= \mathbf{P}(0 \text{ not revisited by time } M)$$

almost surely. But as $M \to \infty$

$$\mathbf{P}(0 \text{ not revisited by time } M) \to \mathbf{P}(0 \text{ is never revisited}),$$

so it follows that

$$\limsup_{n\to\infty} \frac{R_n}{n} \leq \mathbf{P}(0 \text{ is never revisited})$$

and we are done. □

We are now ready to prove part (i) of the Chung-Fuchs theorem, under the additional assumptions that the random walk is on $\mathbb{Z}$ and that $\mathbf{E}X_1$ exists (and therefore $\mathbf{E}X_1 = 0$):

Proof By the strong law of large numbers we know that $\frac{S_n}{n} \to 0$ almost surely. Therefore for every $\epsilon > 0$ we can find n_0 (which is a random variable by itself) such that $\frac{|S_n|}{n} < \epsilon$ for $n > n_0$. Hence for n much larger than n_0 we get that $R_n \leq 3\epsilon n$, so $\limsup \frac{R_n}{n} \leq 3\epsilon$ almost surely.

Since ϵ was arbitrary it follows that $\frac{R_n}{n} \to 0$, so by the Kesten-Spitzer-Whitman theorem $\mathbf{P}(\text{no return to } 0) = 0$, and the walk is recurrent. □

3 Fourier analytic criterion for recurrence

For a random walk on $\mathbb{Z}^d$, let $\varphi(\theta) = \mathbf{E}e^{i\theta\cdot X_1}$ be the characteristic function of X_1. Note that φ is essentially from $\mathbb{T}^d = [-\pi, \pi]^d$ to C.

Reminder

(i) $\mathbf{E}e^{i\theta\cdot S_n} = \varphi^n(\theta)$

(ii) We have the Fourier inversion formula:

$$\mathbf{P}(S_n = y) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-iy\cdot\theta} \varphi^n(\theta) d\theta$$

Theorem 12 *A random walk S_n on $\mathbb{Z}^d$ is reucrrent if and only if*

$$\lim_{r\nearrow 1} \int_{\mathbb{T}^d} \Re\left(\frac{1}{1 - r\varphi(\theta)}\right) d\theta = \infty$$

Remark 13 *(i) It is also true, but more difficult to prove, that S_n is recurrent if and only if*

$$\int_{\mathbb{T}^d} \Re\left(\frac{1}{1 - \varphi(\theta)}\right) d\theta = \infty$$

(ii) Similarly, for general random walks in $\mathbb{R}^d$, S_n is neighbourhood-recurrent if and only if for some $\delta > 0$ (and then for every $\delta > 0$)

$$\int_{(-\delta,\delta)^d} \Re\left(\frac{1}{1 - \varphi(\theta)}\right) d\theta = \infty$$

(this time φ is defined on $\mathbb{R}^d$)

(iii) If X_1 has first and second moments, then (in 1 dimension):

$$\varphi(\theta) = 1 + i\theta\mathbf{E}X_1 - \frac{\theta^2}{2}\mathbf{E}X_1^2 + o(\theta^2)$$

as $\theta \to 0$.

(iv) The weak law of large numbers holds if and only if $\varphi'(0)$ exists, and then $\frac{S_n}{n} \to \mu$ in probability, where $\varphi'(0) = i\mu$ (By a result of E. Pitman)

Proof

$$\mathbf{P}(S_n = 0) = (2\pi)^{-d} \int_{\mathbb{T}^d} \varphi^n(\theta) d\theta,$$

so for $0 < r < 1$ we have

$$\sum_{n=0}^{\infty} r^n \mathbf{P}(S_n = 0) = \sum_{n=0}^{\infty} (2\pi)^{-d} r^n \int_{\mathbb{T}^d} \varphi^n(\theta) d\theta = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{n=0}^{\infty} r^n \varphi^n(\theta) d\theta.$$

Here the final step was Fubini's theorem, which applies because

$$\sum_{n=0}^{\infty} \int_{\mathbb{T}^d} |r^n \varphi^n(\theta)| \, d\theta \leq \sum_{n=0}^{\infty} \int_{\mathbb{T}^d} r^n d\theta < \infty.$$

Since the left hand side is real it follows that

$$\sum_{n=0}^{\infty} r^n \mathbf{P}(S_n = 0) = (2\pi)^{-d} \int_{\mathbb{T}^d} \Re \left(\frac{1}{1 - r\varphi(\theta)} \right) d\theta.$$

When $r \nearrow 1$ the left hand side converges to $\sum_{n=0}^{\infty} \mathbf{P}(S_n = 0)$, so we are done. □

We would like to use this result for a few remarks about the relation between recurrence and moments

Definition 14 *The* symmetric stable random variable *of index α is the random variable X on $\mathbb{R}$ such that*

$$\mathbf{E} e^{i\theta \cdot X} = e^{-|\theta|^\alpha}$$

Such a random variable exists for every $\alpha \leq 2$. If $\alpha = 2$ this is just the gaussian. If $\alpha < 2$ this is a symmetric random variable such that $\mathbf{E}|X|^p < \infty$ for every $p < \alpha$, but $\mathbf{E}|X|^\alpha = \infty$.

Note that

$$\frac{1}{1 - \varphi(\theta)} = \frac{1}{1 - e^{-|\theta|^\alpha}} \sim \frac{1}{|\theta|^\alpha}$$

as $\theta \to 0$. Therefore

$$\int_{(-\delta,\delta)^d} \frac{d\theta}{1 - \varphi(\theta)} = \begin{cases} \infty & : \alpha \geq 1 \\ < \infty & : \alpha < 1 \end{cases},$$

so these random walks are recurrent if and only if $\alpha \geq 1$. For $\alpha > 1$ this is a consequence of the Chung-Fuchs theorem, since $\mathbf{E}X = 0$.

For $\alpha = 1$ X has the Cauchy distribution, with density $\frac{1}{\pi(1+x^2)}$. For such X_1, $\frac{S_n}{n}$ is again Cauchy distributed (as can be seen by calculating its characteristic function), so the weak law of large numbers doesn't hold even though the walk is recurrent.

The recurrence condition is not only about the tail of the random varaible. In 1964, Shepp gave examples of recurrent walks with arbitrary heavy tails. In fact he proved the following:

Theorem 15 *For every function $\epsilon(x)$ such that $\lim_{x\to\infty} \epsilon(x) = 0$, there exists a symmetric random variable X_1 such that*

$$P(|X_1| > x) \geq \epsilon(x)$$

for all large x, and the random walk S_n is neighborhood (and even point) recurrent.

References

[1] Lalley S. One-dimensional random walks. *http://galton.uchicago.edu/ lalley/Courses/312/RW.pdf.*