Random Walks and Brownian Motion | Instructor: Ron Peled
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Lecture 6

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Tags for today's lecture:

This lecture deals with Bounded Harmonic Functions and special σ-fields, the connection with those σ-fields and bounded harmonic functions. We will introduce the concept of Couplings, successful couplings and successful shift couplings, and see the connection between bounded harmonic functions and successful couplings. We will finish the lecture with a little introduction about SRW on Cayley graphs of groups.

1 Harmonic Functions In Context with Markov Chains

Let E be a state space, countable and discrete, ϵ a σ-field.

Define $\mathbb{P} : E \times E \to [0,1]$, $\mathbb{P}(x,y)$ the probability to move from x to y in one step.

$$\sum_{y \in E} \mathbb{P}(y,x) \quad \forall x \in E$$

Definition 1 *$h : E \to \mathbb{R}$ is called Harmonic function if $h(x) = \sum_{y \in E} \mathbb{P}(x,y)h(y)$. in other words: $h(x) = \mathbb{E}^x h(Z_1)$, where $\{(h(Z_n)\}_{n \geq 0}$ is a martingale.*

Definition 2 *A space time harmonic function is a function $h : \times \{0,1,\ldots\} \to \mathbb{R}$ such that $h(x) = \sum_{y \in E} h(y, n+1)$ in other words: $\{(h(Z_n, n)\}_{n \geq 0}$ is a martingale.*

Generalization of Last Class: If $\mathbb{P}$ is irreducible and reccurent then all bounded harmonic functions are constant.

Irreducible: Can get from every $x \in E$ to every $y \in E$ in a finite amount of steps.

Reccurent: $\mathbb{P}^x(Z_n = x \text{ infinitely often}) = 1, \quad \forall x \in E$

on the *Space of Trajectories* $E^\infty = E \times E \times \ldots$ we have *product σ-field* ϵ^∞.

Define $\Theta : E^\infty \to E^\infty$ to be $\Theta(x_0, x_1, \dots) = (x_1, x_2, \dots)$, and

$$\mathcal{F} = \Theta_n^{-1}(\epsilon^\infty)$$

the events not depending on first $n-1$ steps.

Definition 3 *Tail σ-field:*

$$\mathcal{T} := \bigcap_{n=0}^{\infty} \mathcal{F}_n$$

it means that if U is a RV measurable with respect to $\mathcal{T}$ then $\exists functions$ U_n for every n, such that

$$U(x_0, x_1, \dots) = U_n(x_n, x_{n+1}, \dots) \quad \forall n$$

Definition 4 *Invarient σ-field:*

$$\mathcal{I} := \left\{ B \in \epsilon^\infty | B = \Theta^{-1}(B) \right\}$$

it means that if U is a RV measurable with respect to $\mathcal{I}$ then

$$U(x_0, x_1, \dots) = U(x_1, x_2, \dots)$$

Exercise $\mathcal{I} \subseteq \mathcal{T}$.

Examples:

1. SRW on $\mathbb{Z}^d$:
   (a) Visit the origin infinitely often $\in \mathcal{I}$
   (b) Sum of trajectory is infinite $\in \mathcal{I}$
   (c) Parity of initial point $\in \mathcal{T}$ (if $(x_1, \dots, x_d) \in \mathbb{Z}^d$ then parity $= \sum x_i \pmod 2$).
   Since $\mathbb{Z}$ is bipartite, we change parity at every step.
   $\mathbb{P}^\mu$(parity of some initial step is even) may be in $0, 1$ for some μ
2. In a periodic Markov Chain, the initial period $\in \mathcal{T}$.
3. SRW on a completely binary tree:
   (a) Ending up in left subtree $\in \mathcal{I}$.
   (b) parity of initial position $\in \mathcal{T}$.

Definition 5 *Identification Mod 0:*

Say U, V RV's are equal $\mod 0$ *if* $\mathbb{P}^x(U \neq V) = 0 \quad \forall x$

We will only consider $\mathcal{T}$ *and* $\mathcal{I} \mod 0$.

Theorem 6 *(Fundemental relationship between bounded harmonic functions, $\mathcal{T}$ and $\mathcal{I}$)*

1. *There is a bijection between bounded RV mod 0 $U \in \mathcal{T}$ and bounded spacetime harmonic functions given by $U \in \mathcal{T} \Rightarrow h(x,n) = \mathbb{E}^x U_n$. where $U(x_0, \ldots) = U_n(x_n, \ldots)$*
2. *there is a bijection between bounded RV mod 0 $U \in \mathcal{I}$ and bounded harmonic functions given by $U \in \mathcal{I} \rightarrow h(x) = \mathbb{E}^x U$*

Proof

1. $\Rightarrow$

   fix $U \in \mathcal{T}$ define $h(x,n) = \mathbb{E}^x U_n$, h is clearly bounded.

   $$h(x,n) = \mathbb{E}^x U_n = \mathbb{E}^x \mathbb{E}^x (U_n | Z_1) = \mathbb{E}^x \mathbb{E}^{Z_1} U_{n+1} = \mathbb{E}^x h(Z_1, n+1)$$

   so h is spacetime harmonic.

   $\Leftarrow$

   Fix $h(x,n)$ bounded spacetime harmonic function. For every x, $\{h(Z_n, n)\}_{n \geq 0}$ is a bounded martingale hence converges. Define $U(Z_0, Z_1, \ldots)$ as the limit. It is a tail RV isnce it is a limit.

   The mapping is 1:1 since if $h(x,n) = \mathbb{E}^x U_n$ then by the Levy 0-1 law (or martingale convergence theorem for UI martingale) $\lim_{n \to \infty} h(Z_n, n) = U(Z_0, \ldots)$ under $\mathbb{P}^x$.

2. similiar proof.

   notice: $h(x) = \mathbb{E}^x U = \mathbb{E}^x \mathbb{E}^x (U | Z_1) = \mathbb{E}^x \mathbb{E}^{Z_1}(U) = \mathbb{E}^x h(Z_1)$

   $U(Z_0, Z_1, \ldots) = U(Z_1, Z_2, \ldots)$

□

2 Couplings

Definition 7 *For two distributions μ, ν on state spaces S and T, a coupling is a joint distribution on $S \times T$ denoted (X, Y) such that $X \sim \mu$, $Y \sim \nu$.*

We will use this when μ is $\mathbb{P}^x$ on space of trajectories or $\mathbb{P}^\Pi$, and ν is $\mathbb{P}^y$ for $x, y \in E$ Sometimes coupling is used for a coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$ which makes the trajectories collide and continue together.

Under a coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$ let $T = min(n|Z_n^x = Z_n^y)$. If $\mathbb{P}(T < \infty)$, call it a *successful coupling.*

Proposition 8 *If $\forall x, y$ there exist a successful coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$ then $\mathcal{T}$ is trivial, or in other words all bounded harmonic functions are constant.*

$\mathcal{T}$ is trivial: $\forall \mu, A \in \mathcal{T} \quad \mathbb{P}^\mu(A) \in \{0, 1\}$

Proof Fix x, y and let h be a bounded spacetime harmonic function. Need to show $h(x, m) = h(y, m) \quad \forall m$ has a successful couplings of $\mathbb{P}^x$ and $\mathbb{P}^y$.

fix m, notice $\{h(Z_n, n+m)\}_{n \geq 0}$ is a bounded martingale hence converges. In the coupling let the trajectory continue together after colliding.

$$|h(x, m) - h(y, m)| = |\mathbb{E}^x h(Z_n^x, n+m) - \mathbb{E}^y h(Z_n^y, n+m)| =$$
$$\mathbb{E}[h(Z_n^x, n+m) - h(Z_n^y, n+m)]1_{(T>n)}| \leq 2M\mathbb{P}(T > n) \rightarrow_{n\rightarrow\infty} 0$$

where $M = suph$. □

Definition 9 *a successful shift coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$ is a coupling s.t.*

$$\mathbb{P}(\exists n, \exists k, Z_n^x = Z_{n+k}^y) = 1$$

Can ask is this equality holds $\forall n \geq n_0$

Proposition 10 *if $\forall x, y$ there exist a successful shift coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$ then $\mathcal{I}$ is trivial. In other words all bounded harmonic functions are constant.*

Proof fix x, y, h a bounded harmonic function, assume $\mathbb{P}(\forall n \geq N_0, \exists k, Z_n^k = Z_{n+k}^y) = 1$

$$|h(x) - h(y) = |\mathbb{E}[h(z_n^x) - h(z_n^y)]\mathbf{1}_{(N_0>n)} \leq 2M\mathbb{P}(N_0 > n) \rightarrow_{n\rightarrow\infty} 0$$

□

Theorem 11 *the following are equivelant:*

1. *mathcalT is trivial.*
2. *All bounded spacetime harmonic functions are constant.*
3. *there exist a successful coupling* $\forall x, y \in E$
4. *The markov chain is Mixing.*
5. $\left\|\mathbb{P}^{\mu}(Z_n \in \bullet) - \mathbb{P}^{\mu'}(Z_n \in \bullet)\right\|_{TV} \rightarrow_{n\rightarrow\infty} 0 \; \forall \mu, \mu'$ *initial distributions.*
6. $\left\|\mathbb{P}^{\mu}(\theta^n Z \in \bullet) - \mathbb{P}^{\mu'}(\theta^n Z \in \bullet)\right\|_{TV} \rightarrow_{n\rightarrow\infty} 0 \; \forall \mu, \mu'$ *initial distributions.*

$(\|\pi - \pi'\|_T V = 1/2 \sum_x |\pi(x) - \pi'(x)| = \{\text{Max Events A}\}\pi(A) - \pi'(A))$

Theorem 12 *Similarly*

1. *mathcalI is trivial.*
2. *All bounded harmonic functions are constant.*
3. *there exist a successful shift coupling* $\forall x, y \in E$
4. *The markov chain is Cesaro-Mixing.*
5. $\frac{1}{y}\left\|\int_0^t \mathbb{P}^{\mu}(Z\lfloor s\rfloor \in \bullet)ds - \int_0^t \mathbb{P}^{\mu'}(Z\lfloor s\rfloor \in \bullet)ds\right\|_{TV} \rightarrow_{n\rightarrow\infty} 0 \; \forall \mu, \mu'$ *initial distributions.*
6. *analog with averaging*

Proof if T is a successful coupling then $\forall A \subseteq E$ event:

$$\mathbb{P}^{\mu}(Z_n \in A) - \mathbb{P}^{\mu'}(Z_n \in A) = |\mathbb{E}(\mathbf{1}_{(Z_n^{\mu} \in A)} - \mathbf{1}_{(Z_n^{\mu'} \in A)}) \leq \mathbb{P}(T > n)$$

$$\Rightarrow d_{TV}(Z^{\mu} - Z^{\mu'}) \leq \mathbb{P}(T > n)$$

(the coupling inequality). □

Remark 13 *There exist maximal coupling, meaning equality is attained* $\forall n$

Example 14 *on* $\mathbb{Z}^d : \mathcal{T} \supseteq$ *parity of initial point,*

$\mathcal{I}$ *is trivial*

3 Shift Coupling for $\mathbb{P}^x$, $\mathbb{P}^y$

1. Case 1: All coordinates difference of x and y are even. In this case we will have a successful coupling. Makes the walks always move on the same coordinate. If they are equal on that coordinate move together, if not, move independently. By recurrence of 1D SRW this is a successful coupling.

2. Case 2: Move only first walk until add coordinates differences are equal. Countinue as Case 1.

Remark 15 *1. On $\mathbb{Z}^d$ there are on non-constant harmonic functions which are*

   (a) non negative, or

   (b) sublinear.

2. $\mathcal{T} = \sigma\{$Parity of initial point is even$\}$. Left as an Exercise.

Example 16 *An irreducible recurrent chain with all states of period d (meaning: gcd(return possible times)=d)*

$$\Rightarrow \mathcal{T} = \sigma\{\text{the } d \text{ possible periodic classes for starting point}\}$$

Example 17 *Long range RW on $\mathbb{Z}$ (ornsteins coupling).*

Proposition 18 *If S_n is a RW on $\mathbb{Z}$, irreducible and aperiodic, then $\mathcal{T}$ is trivial.*

Proof fix $x, y \in \mathbb{Z}$ need to find a successful coupling.

Take M large enough so that walk coordinates to have only steps of Magnitude $\leq M$ is still ireducible and aperiodic. Have the two walks take all jumps $\geq M$ together. Small jumps are done independently.

We have now coupled S_n^x and S_n^y. Notice $S_n^x - S_n^y$ is a RW with a bounded increment distribution and mean 0, so $S_n^x - S_n^y$ is recurrent. It starts at $x - y$ and by aperiodicity and irreducibility it has all of $\mathbb{Z}$ as possible values, hence $\mathbb{P}(S_n^x - S_n^y \text{ i.o.}) = 1$ □

Example 19 *RW on Regular tree: It is possible to prove that $\mathcal{I} = \sigma\{$ending brance$\}$, $\mathcal{T} = \sigma\{\mathcal{I}, parity\}$*

4 SRW on Cayley graphs of groups

G a group, S set of generators.

Open Question: Is Liouville is a group property?

Remark 20 *Liouville is not preserved under rough isometries (quasi-isometries)*

Theorem 21

1. *Liouville* $\Leftrightarrow$ *for SRW* $\lim_{n\to\infty} \frac{d(S_n, \textit{starting point})}{n} = 0$
2. *Kaimanovich-Vershik: Liouville* $\Leftrightarrow \frac{\textit{entropy of } S_n}{n} \to_{n\to\infty} 0$
   $\textit{entropy}(X) = -\sum_x \mathbb{P}(X = x) Log(\mathbb{P}(X = x))$

Example 22 *Lamplighter Group for* $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, G *group.*

Cayley Graph parameterized by: $\{g, f : G \to \{0, 1\}\}$

where: g *- position of lamplighter,*

f *- state of lamps*

Proposition 23 *transient* $G \Rightarrow$ *Lamplighter on* G *is not Liouville.*

Proof Lamp on origin remains lt from some point on $\in \mathcal{I}$ $\square$

Proposition 24 *recurrent* $G \Rightarrow$ *Lamplighter is Liouville.*

Left as an exercies.

References