Random Walks and Brownian Motion Tel Aviv University Spring 2011	Instructor: Ron Peled

Lecture 7

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The following lecture (and the next one) will be an introduction to modern *Potential Theory*. We will see how the connection between Simple Random Walks and Harmonic Functions is used to solve problems in difference and differential equations with probabilistic tools. Our main topics are: The *Dirichlet* problem, *Poisson's* equation, and *Green's* functions. We will use tools that were obtained during previous lectures, such as the LCLT, the optimal stopping theorem, and the martingle convergence theorem. Today's lecture is covered by [1].

Tags for today's lecture: Laplacian, Dirichlet problem, Maximum principle, Poisson kernel, Poisson equation, Green's function, Local central limit theorem for higher dimensions.

1 Notations

We discuss functions of the class $f : \mathbb{Z}^d \to \mathbb{R}$

Definition 1 *The discrete Laplacian or the generator of a SRW:*

$$(\mathcal{L}f)(x) = E^x[f(S_1) - f(x)] = \frac{1}{2d}\sum_{x\sim y} f(y) - f(x)$$

Remark 2 *f is harmonic $\Leftrightarrow \mathcal{L}f \equiv 0$, f is sub-harmonic $\Leftrightarrow \mathcal{L}f0$*

Denote $A \subset \mathbb{Z}^d$.

Definition 3 *The exit time from A is defined as $\tau_A = min(n \geq 0 : S_n \notin A)$*

Definition 4 *The outer boundary of A is $\partial A = \left\{y \in \mathbb{Z}^d \setminus A : \exists x \in A, y \sim x\right\}$*

Definition 5 *The closure of A is $\bar{A} = A \cup \partial A$*

Remark 6 *If $f : \bar{A} \to \mathbb{R}$, then $\mathcal{L}f$ is well defined in A.*

2 Dirichlet Problem

The *Dirichlet* problem for $\emptyset \neq A \subset \mathbb{Z}^d$ given some boundary information $F : \partial A \to \mathbb{R}$ is to find $f : \bar{A} \to \mathbb{R}$ such that:

1. $\forall x \in A,\ (\mathcal{L}f)(x) = 0$, that is, f is harmonic in A
2. $\forall x \in \partial A,\ f(x) = F(x)$

Theorem 7 *Dirichlet problem 1*
Suppose $\forall x \in A\ \mathbf{P}^x(\tau_A < \infty) = 1$ and F is bounded, then there is a unique bounded solution to the Dirichlet *problem, and it is given by*

$$f(x) = \mathbb{E}^x F(S_{\tau_A}) \tag{1}$$

This gives a connection between random walks and difference equation problems. In the continuous case one can do the same with differential equations and Brownian motion.

Proof It is easy to check that (1) defines a bounded solution. Suppose f is a bounded solution, we need to show that f is given by (1).
Note that $\{f(S_{n\wedge\tau_A})\}_{n\geq 0}$ is a bounded martingale because f is harmonic on A and constant outside A. So

$$f(x) = \mathbb{E}^x f(S_{n\wedge\tau_A}) = \mathbb{E}^x f(S_{\tau_A}) = \mathbb{E}^x F(S_{\tau_A})$$

by the optional stopping theorem and because f and F agree on the boundary.

□

Remark 8 *Under the assumptions of the theorem we get the* maximum principle *for harmonic functions on on A:*

$$\sup_{x\in\bar{A}} |f(x)| = \sup_{x\in\partial A} |f(x)|$$

Remark 9 *If A is finite then all functions on ∂A are bounded and we can show existence and uniqueness without further assumptions. Furthermore, existence and uniqueness can be proven by linear algebra by noting that we have $|A|$ equations and $|A|$ unknowns, but this doesn't give the probablistic presentation of the solution.*

Remark 10 *If A is infinite the problem can have infinite unbounded solutions. For example, for $A = \{1, 2, 3, ...\}, F(0) = 0, d = 1$ we can take any linear function $f(x) = bx,\ b \in \mathbb{R}$.*

Remark 11 *If $d \geq 3$ we can have that $\mathbf{P}^x(\tau_A < \infty) < 1$ in which case we can define a bounded solution to $F \equiv 0$ by $f(x) = \mathbf{P}^x(\tau_A = \infty)$. Therefore, the condition $\mathbf{P}^x(\tau_A < \infty) = 1$ is required for uniqueness.*

Continuous analogue Suppose $D = \{x \in \mathbb{R}^d : |x| < 1\}$ or some other "nice" domain and some $f : \bar{D} \to \mathbb{R}$ such that f is continuous on $\bar{D}$ and C^2 on D and $\triangle f \equiv 0$ on D. Then

$$f(x) = \mathbb{E}^x f(B_\tau)$$

where B is a Brownian motion and τ is the exit time from D.
Furthermore, B_τ has a density with respect to the *Lebesgue measure* on ∂D. This density is called the *Poisson kernel* and it is given by

$$H(x,y) = C_d \frac{1 - |x|^2}{|x - y|^d} \tag{2}$$

which is the density at y for B_τ that started at x.
If you know H then you can get f by

$$f(x) = \int_{\partial D} H(x,y) f(y) ds(y)$$

where s denotes surface measure.
This relation in the origin for the *Poisson boundary* that was discussed in lecture 5. This explicit form of H has two consencuences:

Fact 12 *Derivative estimates*
f is smooth in D and its k'th order derivaties are bounded, such that

$$f^{(k)}(x) \leq c_k \|f\|_\infty$$

for some c_k independent of f ($\|f\|_\infty$ is attained on ∂D).

Fact 13 *Harnack inequality*
For any $0 \leq r \leq 1$, $\exists C_r$ independent of f such that if $f \geq 0$ then

$$\max_{\|x\|<r} f(x) \leq c_r \min_{\|x\|<r} f(x)$$

Similar statements are true in a discrete setting but we will not discuss them.

The following theorem addresses the case where the RW exits the set in an infinite time.

Theorem 14 *Dirichlet problem 2*
Suppose F is a bounded function on ∂A, then the only bounded solutions to the Dirichlet *problem are of the form*

$$f(x) = \mathbb{E}^x F(S_{\tau_A}) 1_{\tau_A < \infty} + b\mathbf{P}^x(\tau_A = \infty) \tag{3}$$

for some $b \in \mathbb{R}$.
Therefore, the space of bounded solutions is 2-dimensional.

Proof It will be convinient to use the *lazy* RW, $\tilde{S}_n$, which has a chance $1/2$ not to move at every time step and a chance $1/2$ to move like a SRW. We will use what was shown in the previous class - that $\forall x, y \in \mathbb{Z}^d$ there exists a successful coupling of $\tilde{S}_n^x$ and $\tilde{S}_n^y$ (there are no parity issues because the *lazy* RW is aperiodic).
Assume $\exists x \in A$ such that $\mathbf{P}^x(\tau_A = \infty) > 0$, otherwise the previous theorem applies. Let f be a bounded solution to the *Dirichlet* problem. Note that $\{f(\tilde{S}_{n\wedge\tau_A})\}_{n\geq 0}$ is a bounded martingale. Hence,

$$f(x) = \mathbb{E}^x[f(\tilde{S}_{n\wedge\tau_A})] = \mathbb{E}^x[f(\tilde{S}_n)] - \mathbb{E}^x[1_{\tau_A<n}(f(\tilde{S}_n) - f(\tilde{S}_{\tau_A})] \tag{4}$$

by extending f to $\mathbb{Z}^d$ in any bounded way. Now notice that $\forall x, y \in A$ we have

$$|\mathbb{E}[f(\tilde{S}_n^x) - f(\tilde{S}_n^y)]| \leq 2\|f\|_\infty \mathbf{P}(T > n) \xrightarrow[n\to\infty]{} 0 \tag{5}$$

where T is the coupling time. The inequality is the bound by the coupling and the limit is given because the coupling is successful. Thus, by (4) and (5) we get

$$|f(x) - f(y)| \leq 2\|f\|_\infty [\mathbf{P}^x(\tau_A < \infty) + \mathbf{P}^y(\tau_A < \infty)]$$

Denote $U_\epsilon = \{x \in A : \mathbf{P}^x(\tau_A = \infty) \geq 1 - \epsilon\}$ the set of points that have a "good" chance to remain in A forever.

Exercise $\forall \epsilon > 0, U_\epsilon \neq \emptyset$

Thus, $\forall x, y \in U_\epsilon$ we get $|f(x) - f(y)| \leq 4\epsilon\|f\|_\infty$. Taking any sequence $\{x_n\} \in U_\epsilon$ the limit $\lim_{n\to\infty} f(x_n)$ exists. Denote this limit by b, then

$$|f(x) - b| \leq 4\epsilon \|f\|_\infty \quad \forall \epsilon > 0, x \in U_\epsilon \tag{6}$$

Next, define $\rho_\epsilon = \min\{n : \tilde{S}_n \in U_\epsilon\}$.

Exercise $\forall x \in A, \mathbf{P}^x(\tau_A \wedge \rho_\epsilon < \infty) = 1$

Therefore, the optional stopping theorem implies

$$f(x) = \mathbb{E}^x f(\tilde{S}_{\tau_A \wedge \rho_\epsilon}) = \mathbb{E}^x F(\tilde{S}_{\tau_A}) 1_{\tau_A \leq \rho_\epsilon} + \mathbb{E}^x f(\tilde{S}_{\rho_\epsilon}) 1_{\rho_\epsilon < \tau_A}$$

We are almost there. $\rho_\epsilon \xrightarrow[\epsilon \to 0]{} \infty$, therefore $1_{\rho_\epsilon \leq \tau_A} \xrightarrow[\epsilon \to 0]{} 1_{\tau_A = \infty}$ and with (6) we get:

$$\lim_{\epsilon \to 0} \mathbb{E}^x f(\tilde{S}_{\rho_\epsilon}) 1_{\rho_\epsilon \leq \tau_A} = b \mathbf{P}^x(\tau_A = \infty)$$

Now, by the dominated convergence theorem,

$$\lim_{\epsilon \to 0} \mathbb{E}^x F(\tilde{S}_{\tau_A}) 1_{\tau_A < \rho_\epsilon} = \mathbb{E}^x F(\tilde{S}_{\tau_A}) 1_{\tau_A < \infty}$$

and we have all we need, as $F(\tilde{S}_{\tau_A}) 1_{\tau_A < \infty} \stackrel{d}{=} F(S_{\tau_A}) 1_{\tau_A < \infty}$ $\square$

Remark 15 *We can write $H_A(x, y) = \mathbf{P}^x(\tau_A < \infty, S_{\tau_A} = y)$ as the chance of exiting at y when starting at x. This is called the* harmonic measure. *We can add that $H_A(x, \infty) = \mathbf{P}^x(\tau_A = \infty)$. Finally, if $F(\infty) = b$, then we have shown that*

$$f(x) = \sum_{y \in \partial A \cup \infty} H_A(x, y) F(y) \tag{7}$$

which is analogougous to the Poisson kernel *in* (2)*. The fact that it has just one point at ∞ is the fact that $\mathbb{Z}^d$ has a trivial* Poisson *boundary.*

Remark 16 *$H_A(x, y)$ is the solution to the* Dirichlet *problem with*

$$F(z) = \begin{cases} 1, & z = y \\ 0, & z \neq y \end{cases}$$

This is a generalization to the Gambler's Ruin *problem in 1-dimension.*

3 Poisson's Equation

The following is *Poisson's equation* for f, given a set A and a function $g : A \to \mathbb{R}$:

1. $(\mathcal{L}f)(x) = -g(x)$
2. $f(x) = 0 \;\; \forall x \in \partial A$

Theorem 17 *Suppose $\emptyset \neq A \subset \mathbb{Z}^d$ with $\mathbf{P}^x(\tau_A < \infty) = 1 \forall x \in A$ and g has a finite support (which is like taking a finite set A), then there is a unique solution to* Poisson's *equation, and it is given by:*

$$f(x) = \mathbb{E}^x \sum_{j=0}^{\tau_A - 1} g(S_j) \tag{8}$$

Sketch of Proof This proof is left as an exercise. Similar to the proof of (7), verify that f is a solution. Then assume $h(x)$ is also a solution and check that $(\mathcal{L}(h-f))(x) = 0$ and $h(x) - f(x) = 0$ on ∂A. Therefore, it follows from (1) that $h \equiv f$. $\square$

Remark 18 *Define the* Green's *function in A as*

$$G_A(x,y) = \mathbb{E}^x\left[\# \text{ visits to } y \text{ before leaving } A \text{ when starting at } x\,\right] \tag{9}$$

Then $f(x) = \sum_{y \in A} G_A(x,y) g(y)$.

Remark 19 *If we take $g \equiv 1$ on a finite A, we get that $f(x) = \mathbb{E}^x[\tau_A]$. Therefore the expected exit time when starting from x can be found by calculating $\sum_{y \in A} G_A(x,y)$.*

Remark 20 *On a finite A, $G_A(x,y)$ is a $|A| \times |A|$ matrix. Similarity,*

$$\mathcal{L}_A(x,y) = \begin{cases} -1, & y = x \\ \frac{1}{2d}, & y \sim x \\ 0, & \text{otherwise} \end{cases}$$

is also a $|A| \times |A|$ matrix. The theorem tells us that

$$G_A = (-\mathcal{L}_A)^{-1} \tag{10}$$

4 The Local Limit Theorem

Recall that in 1-dimension, S_n is close to $N(0,n)$ and we saw (Lecture 3) that if $x = o(n^{3/4})$ and $x+n$ is even, then:

$$P(S_n = x) \sim 2\frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}$$

as $n \to \infty$.
Next, we present the analogous for higher dimensions: for $d \geq 1$, S_n is close to $N(0, \frac{1}{d}\mathrm{I}n)$, where $\frac{1}{d}\mathrm{I}$ is the covariance matrix of one step of the SRW:

$$\frac{1}{d}\mathrm{I} = \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} = (\mathbb{E}\left[X_{1,i}, X_{1,j}\right])_{i,j}$$

The density of $N(0, \frac{1}{d}\mathrm{I}n)$ is $(\frac{d}{2\pi n})^{\frac{d}{2}}e^{-\frac{d|x|^2}{2n}}$ and therefore we expect that if $x+n$ is even (x and n have the same parity), then

$$\mathbf{P}(S_n = x) \sim 2(\frac{d}{2\pi n})^{\frac{d}{2}}e^{-\frac{d|x|^2}{2n}}$$

Indeed, if we define the *lhs* as $P_n(x)$ and *rhs* as $\bar{P}_n(x)$, the error is given by

$$E_{n,x} = \begin{cases} |P_n(x) - \bar{P}_n(x)|, & x+n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

and we get the following theorem:

Theorem 21 *LCLT*
Given the above notations,

$$\sup_x E_{n,x} = O(n^{-\frac{d+2}{2}})$$

as $n \to \infty$.

Sketch of Proof There are two main approaches:

1. In SRW in $\mathbb{Z}^d$ the number of steps taken in each coordinate after n steps overall has a multinomial distribution. We need to establish a LCLT for this distribution using *Stirling's Formula*. Then we can use the 1-dimensional theorem to get the d-dimensional theorem, as there are approximately n/d steps in every coordinate.

2. A more general approach: if S_n and $\bar{S}_n$ are two distributions on $\mathbb{Z}^d$, then the respective *Fourier transforms* are

$$f(\lambda) = \mathbb{E}e^{i\lambda S_n},\ \bar{f}(\lambda) = \mathbb{E}e^{i\lambda \bar{S}_n},\ \lambda \in [-\pi,\pi]^d$$

Then, by using the *inverse Fourier transform*:

$$\sup_x |\mathbf{P}(S_n = x) - \mathbf{P}(\bar{S}_n = x)| \leq \int_{[-\pi,\pi]^d} |f(\lambda) - \bar{f}(\lambda) d\lambda$$

or by using the L_p norm notation:

$$\|\mathbf{P}(S_n = x) - \mathbf{P}(\bar{S}_n = x)\|_\infty \leq \|f - \bar{f}\|_1$$

Very roughly, to control $\|f - \bar{f}\|_1$, f and $\bar{f}$ are very small when λ is far from zero, and by a *Taylor expansion*, $f - \bar{f}$ is small when λ is close to zero.
More details on pg. 3-4 in [2].

□

Proposition 22 *(Large Deviation) There exist $C_d, c_d > 0$, dependent only on the dimension d, such that*

$$\mathbf{P}(|S_n| > r\sqrt{n}) \leq C_d e^{-c_d r^2}$$

Proof In 1-dimension we proved this (Lecture 3). In higher dimensions it follows by the *union bound*:
If $|S_n| > r\sqrt{n}$ then at least one coordinate must be $> r\sqrt{n}$. □

5 Green's Functions

There are three types of *Green's functions*:

1. For a domain $A \subset \mathbb{Z}^d$ we define:

$$G_A(x,y) = \mathbb{E}^x\,[\text{\# visits to y when starting at x before leaving A}]$$

This function was discussed in (9) where it was shown to solve *Poisson's equation.*

2. For the entire space we define:

$$G(x,y) = \mathbb{E}^x\left[\#\text{ visits to y when starting at x}\right] = \sum_{n=0}^{\infty} \mathbf{P}^x(S_n = y) \tag{11}$$

This function can be written with one argument as:

$$G(x,y) = G(y-x) = \mathbb{E}\left[\#\text{ visits to y when starting at 0}\right]$$

The funciton is well-defined only when $d \geq 3$ as it requires transience.
For $d = 1, 2$, the *potential kernel*:

$$a(x) = \sum_{n=0}^{\infty} \left[\mathbf{P}(S_n = 0) - \mathbf{P}(S_n = x)\right] = \text{"}G(0) - G(x)\text{"} \tag{12}$$

plays the same role as *Green's function*.
As in (10), we have $G = (-\mathcal{L}_{\mathbb{Z}^d})^{-1}$.

3. The *Geometrically Killed Green's Function*
For $0 < \lambda < 1$, let $T \sim Geo(1-\lambda)$ be the killing time of the RW, meaning that in each step, the RW has a chance λ to survive and a chance $1-\lambda$ to be killed. Then we can define:

$$\begin{aligned} G(x,y;\lambda) &= \mathbb{E}^x\left[\#\text{ visits to y when starting at x before time T}\right] = \\ &= \sum_{n=0}^{\infty} \lambda^n \mathbf{P}^x(S_n = y) \end{aligned} \tag{13}$$

Again, this function can be written with one argument as

$$\begin{aligned} G(x,y;\lambda) &= G(y-x;\lambda) = \\ &= \mathbb{E}\left[\#\text{ visits to y when starting at 0 before time T}\right] \end{aligned}$$

Geometrically killed walks are useful in modeling a walk that does roughly n steps. A geometrically killed walk with $\lambda = 1 - \frac{1}{n}$ will have $\mathbb{E}[T] = n$, but it will keep the *Markovian property*, as oppsed to a walk that does *exactly* n steps.

We can deduce about the asymptotics of *Green's functions* from the *LCLT*. Consider a geometrically killed walk. Denote R as the number of distinct values seen before the walk is killed. If $\lambda = 1 - \frac{1}{n}$, then R is roughly the number of distinct values seen in n steps.
More generally:

Remark 23 *$\mathbb{E}[R] = \mathbb{E}[T]\,\mathbf{P}$(no return to zero before killed), where T is the killing time.*

This is similar to the *KSW theorem*. The proof is also by the *Birkhoff ergodic theorem*.

Note that $\mathbf{P}$(no return to zero before killed) $= G(0;\lambda)^{-1}$, because the number of visits to zero is geometrically distributed.
Finally, by the LCLT

$$G(0;\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{P}(S_n = 0) = \sum_{n=0}^{\infty} \lambda^n C_d \frac{1}{n^{d/2}} + O(n^{-\frac{d+2}{2}})$$

and as $\lambda \to 1$ we get

$$G(0;\lambda) = C_d F(\frac{1}{1-\lambda})$$

where F is given by

$$F(s) = \begin{cases} \sqrt{s}, & d = 1 \\ \log s, & d = 2 \end{cases}$$

Thus, if $\lambda = 1 - \frac{1}{n}$ we have

$$G(0;\lambda) \sim \begin{cases} c_1\sqrt{n}, & d = 1 \\ c_2 \log n, & d = 2 \end{cases}$$

where c_1 and c_2 are explicit constants.
Therefore,

$$\mathbb{E}[R] = \begin{cases} \frac{\sqrt{n}}{c_1}, & d = 1 \\ \frac{n}{c_2 \log n}, & d = 2 \end{cases}$$

And with this we end this class.

References

[1] Lawler, G. and Limic V. (2010). Random Walk: A Modern Introduction, Cambridge University Press.

[2] Lawler, G.F. (1996). Intersections of Random Walks, Birkhäuser.