Lecture 8

Lecture date: Apr 29, 2011 *Scribe: Uri Grupel*

In this lecture we compute asumptotics estimates for the Green's function and apply it to the exiting annuli problem. Also we define Capacity, Polar set Prove the B-P-P theorem of Martin's Capacity for Markov chains[3] and use apply it on the intersection of RW problem.

Tags for today's lecture: Green's function, exiting annuli, Capacity, Polar set, Martin Capacity, intersection of RW.

1 Asymptotics for Green's funcion

Here we show the Asymptotics for Green's funcion and application for exiting annuli in dimension $d \geq 3$
Reminders:

- Local Centeral Limit Theorem (LCLT):

$$\sup_{x\in\mathbb{Z}^d \; x \; and \; n \; of \; same \; parity} \left| \mathbf{P}(S_n = x) - 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{2n}} \right| = O(n^{-\frac{d+2}{2}})$$

  as $n \to \infty$
  We'll denote $\bar{p}(n,x) = 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{2n}}$ and $E(n,x) = \left|\mathbf{P}(S_n = x) - 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{2n}}\right|$

- Large Deviation: $\exists C_d, c_d > 0$ such that $\forall r, n$ $\mathbf{P}(|S_n| \geq r\sqrt{n}) \leq C_d e^{-c_d r^2}$

- Green's Function: In dimension $d \geq 3$ the Green's functions is defined by

$$G(x,y) = G(x-y) = \mathbf{E}^x(number \; of \; vists \; to \; y) = \sum_{n=1}^{\infty} \mathbf{P}^x(S_n = y)$$

  Since in $d = 1,2$ this sum is always ∞ In dimesion $d = 1,2$ the ptential kernel plays a similar role

$$a(x) = \sum_{n=1}^{\infty} \mathbf{P}(S_n = 0) - \mathbf{P}(S_n = x) = "'G(0) - G(x)"'$$

  The use of quation marks is due to the fact that in dimension $d = 1,2$ the sum in the Green's function definition would be ∞

Theorem 1 *For $d \geq 3$, $G(x) \approx a_d|x|^{2-d}$ as $|x| \to \infty$ where $a_d = \frac{d}{2}\Gamma\left(\frac{d}{2}-1\right)\pi^{-\frac{d}{2}} = \frac{2}{(d-2)w_d}$ and $w_d = vol(B^d)$ the volume of the unit ball in $\mathbb{R}^d$.*
More precisley $\forall \alpha < d$, $G(x) = a_d|x|^{2-d} + o(|x|^{-\alpha})$ as $|x| \to \infty$

ProofFix $x \neq 0$ of even parity, $G(x) = \mathbf{E}(number\ of\ visits\ to\ x) = \sum_{n=1}^{\infty} \mathbf{P}(S_n = x)$. *Note first*

$$\sum_{n=1}^{\infty} \bar{p}(2n,x) = \sum_{n=1}^{\infty} 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{4n}} \underset{*}{=} \int_0^{\infty} 2\left(\frac{d}{2\pi t}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{4t}} + O(|x|^{-d}) =$$

$$= \frac{d}{2}\Gamma\left(\frac{d}{2}-1\right)\pi^{-\frac{d}{2}}|x|^{2-d} + O(|x|^{-d})$$

Where $O(|x|^{-d})$ is with respect to $|x| \to \infty$. Note that the estimate in $*$ is according to the order of the first term for $|x| \sim n$
Now $G(x) = \sum_{n=1}^{\infty} \bar{p}(2n,x) + E(2n,x)$ we need only to show that for any $\alpha < d$ $\sum_{n=1}^{\infty} E(2n,x) = o(|x|^{-\alpha})$, to see this we pick a threshold n_0 close to $|x|^2$ E.g, $n_0 = \frac{c|x|^2}{\sqrt{\log|x|}}$, then write

$$\sum_{n=1}^{\infty} E(2n,x) = \underbrace{\sum_{n=1}^{\lfloor n_0 \rfloor} E(2n,x)}_{By\ large\ deviation\ and\ choice\ of\ c\ = O(|x|^{-d})} + \underbrace{\sum_{\lfloor n_0 \rfloor + 1}^{\infty} E(2n,x)}_{By\ LCLT\ = O(n_0^{-d}) = o(|x|^{-\alpha})\ for\ any\ \alpha<d}$$

For x of odd parity note $G(x)$ is harmonic for $x \neq 0$, so $G(x) = \frac{1}{2d}\sum_{i=1}^{2d} G(x+e_i)$ (where e_i are the unit vectors) giving the result.

Remark 2
1. *In the continuem, for Brownien Motion in $\mathbb{R}^d$ ($d \geq 3$) the Green function is exactly $a_d|x|^{2-d}$, this function is harmonic except at $x = 0$ and has laplacian $-\delta_0$ at 0.*
2. *The error in the previus theorem is even $O(|x|^{-d})$*

Application (exiting annuli or quantitative transience)
In $d \geq 3$ denote $A_{n,m} = \{x \in \mathbb{Z}^d;\ n < |x| < m\}$, $\tau_{n,m} = \min\{j;\ S_j \notin A_{n,m}\}$, then for $x \in A_{n,m}$

$$\mathbf{P}^x(|S_\tau| \leq n) = \frac{|x|^{2-d} - m^{2-d} + O(n^{1-d})}{n^{2-d} - m^{2-d}}$$

In particular taking $m \to \infty$, we get $\mathbf{P}^x(\exists j;\ |S_j| \leq n) = \frac{|x|^{2-d}}{n^{2-d}} + o(n^{-1})$

ProofSince $G(x)$ is harmonic except in $x = 0$, $M_j := G(S_j \wedge \tau)$ is a bounded martingale so

$$\underbrace{G(x)}_{\approx|x|^{2-d}} = \mathbf{E}^x M_j \underset{optional\ sampeling}{=}$$

$$= \mathbf{P}(|S_\tau| \le n) \underbrace{\mathbf{E}^x(G(S_\tau)|\ |S_\tau| \le n)}_{\approx n^{2-d}} + (1 - \mathbf{P}(|S_\tau| \le n)) \underbrace{\mathbf{E}^x(G(S_\tau)|\ |S_\tau| \ge m)}_{\approx m^{2-d}}$$

Noting that $G(x) = |x|^{2-d} + O(|x|^{1-d})$ *(weaker than the error in the theorem). By isolation* $\mathbf{P}(|S_\tau| \le n)$ *we get the result. Similarly one can prove*

Proposition 3 *In* $d \ge 3$ *letting* $C_n = \{x \in \mathbb{Z}^d;\ |x| < n\}$ *and* $\tau = \tau_n = min\{j;\ S_j \notin C_n or S_j = 0\}$ *for* $x \in C_n$

$$\mathbf{P}(S_\tau = 0) = \frac{a_d}{G(0)}(|x|^{2-d} - n^{2-d}) + O(|x|^{1-d})$$

as $|x| \to \infty$

Reminder: *(Second type of Green's function) For any* $A \subseteq \mathbb{Z}^d$, $G_A(x,y) = \mathbf{E}^x(number\ of\ visits\ to\ y\ before\ exiting\ A)$

Question 4 *How do we estimate* $G_a(x,y)$*?*

Definition 5 *For* $A \subseteq \mathbb{Z}^d$ *we define the boundry of* A *as* $\partial A = \{x \in \mathbb{Z}^d;\ x \notin A,\ x \sim y\ for\ some\ y \in A\}$

Proposition 6 *For finite* $A \subseteq \mathbb{Z}^d$ $\forall x, z \in A$

$$G_A(x,z) = G(x,z) - \sum_{y \in \partial A} H_A(x,y) G(y,z)$$

where $H_A(x,y) = \mathbf{P}^x(y\ is\ first\ exit\ point\ of\ A)$

Proof*Let* $\tau = min\{j;\ S_j \notin A\}$ *then*

$$G_A(x,z) = \mathbf{E}^x\left(\sum_{j=0}^{\tau-1} 1_{S_j=z}\right) = \mathbf{E}^x\left(\sum_{j=0}^{\infty} 1_{S_j=z} - \sum_{j=\tau}^{\infty} 1_{S_j=z}\right)$$

The first term is $G(x,z)$*, and the second is the same as in the proposition.*

Combining proposition 3 and proposition 6 we have the following proposiotion

Proposition 7 $G_{C_n}(x,0) = a_d(|x|^{2-d} - n^{2-d}) + O(|x|^{1-d})$ ***Proof****Exercise.*

Analogous results for dimension $d = 1, 2$ *[1][2]*

- *In* $d = 1$*:* $a(x) = |x|$.
  In $d = 2$*:* $\exists k$ *such that* $\forall \alpha < 2$

$$a(x) = \frac{2}{\pi} \log |x| + k + o(|x|^{-\alpha})$$

  With $k = \frac{2\gamma}{3} + \frac{3}{\pi} \log 2$ *where* $\gamma = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j} - \log n$ *the Eulrt constant.*

- $G_A(x,z) = \sum_{y \in \partial A} H_A(x,y) a(y-x) - a(z-x)$ *for some finite* A *and* $x, y \in A$

- *For* $d = 2$ *In the annulus* $A_{n,m}$, $\forall x \in A_{n,m}$

$$\mathbf{P}^x(|S_\tau \leq n) = \frac{\log m - \log |x| + O(n^{-1})}{\log m - \log n}$$

  (Quantitative recurrence)

- *For* $d = 2$ $x \in C_n$, $\forall \alpha < 2$

$$G_{C_n}(x,0) = \frac{2}{\pi}(\log n - \log |x|) + o(|x|^{-\alpha}) + O(n^{-1})$$

2 Capacity

In this section we define Capacity and Polar sets, intoduce Kakutanis theorem charactrizing polar sets, and prove the Benjamini-Pemantle-Peres theorem which gives a quantitive connection between the Capacity of a set and a Markov Chain on the set. In the end of the section we see some application to the intersection of random walks. This section follow [3]

Definition 8 *Given a measure space* $(\Lambda, \mathcal{F})$*, a measurable function* $F : \Lambda \times \Lambda \to [0, \infty)$ *(kernel) and a finite measure* μ *of* Λ*, the underlineF - energy of* μ *with respect to* F *is*

$$I_F(\mu) = \int_\Lambda \int_\Lambda F(x,y) d\mu(x) d\mu(y)$$

Remark 9 *It is useful to think of* $\Lambda \subseteq \mathbb{R}^3$*,* μ *charge density of a certain material and* $F(x,y) = |x-y|^{-1}$

Definition 10 *The capacity of* Λ *with respect to* F *is*

$$cap_F(\Lambda) = \left(\inf_\mu I_F(\mu) \right)^{-1}$$

Where μ *is a probability measure on* Λ*, and* $\infty^{-1} = 0$.

Remark 11 *Capacity is monotone increasing in the set. It is a measure of the size of* λ *with respect to* F.

Remark 12 *For is* Λ *will be a countable with* $\mathcal{F} = \{all\ subsets\ of\ \Lambda\}$
Sometimes we'll also mention $\Lambda \subseteq \mathbb{R}^d$ *with* $\mathcal{F}$ *Borel* σ*-Field.*

Kakutani's theorem *(1944):*

Definition 13 *A Borel* $A \subseteq \mathbb{R}^d$ *is called Polar if* $\mathbf{P}^x(\exists t > 0, B(t) \in A) = 0\ \forall x \in \mathbb{R}^d$ *for a Brownian Motion* $B(t)$

Theorem 14 *(Kakutani) A Borel* $A \subseteq \mathbb{R}^d$ *is Polar if and only if* $cap_F(A) = 0$ *for*

$$F(x,y) = \begin{cases} \left|\log\left(\frac{1}{|x-y|}\right)\right| & d = 2 \\ \frac{1}{|x-y|^{d-2}} & d \geq 3 \end{cases}$$

Now let $\{X_n\}$ *be a Markov chain on a countable set* Y, *with transition probability* $p(x,y)$. *Let*

$$G(x,y) = \mathbf{E}^x(\#\ visits\ to\ y) = \sum_{n=0}^{\infty} p^{(n)}(x,y)$$

Theorem 15 *(Benjamini, Pemantle, Peres) If* $\{X_n\}$ *is a transient chain (any state is visited onlt finitly many times a.s) then for all starting point* $\rho \in Y$ *and any* $\Lambda \subseteq Y$

$$\frac{1}{2} cap_K(\Lambda) \leq \mathbf{P}^\rho(\exists n;\ X_n \in \Lambda) \leq cap_K(\Lambda)$$

where K *is the Martin kernel* $K(x,y) = \frac{G(x,y)}{G(\rho,y)}$. *Furthermore, letting* $cap_K^{(\infty)} := \inf_{\Lambda_0\ finite} cap_K(\Lambda \setminus \Lambda_0)$

$$\frac{1}{2} cap_K^{(\infty)}(\Lambda) \leq \mathbf{P}(X_n \in \Lambda\ i.o) \leq cap_K^{(\infty)}(\Lambda)$$

Proof *To get upper bound it is enough to find one* μ. *Let* $\tau = \min\{n;\ S_n \in \Lambda\}$ *(*$\tau = \infty$ *if* Λ *is not hit), for* $x \in \Lambda$ $\nu(x) = \mathbf{P}^\rho(\tau < \infty,\ S_\tau = x)$ $\nu(\Lambda) = \mathbf{P}^\rho(\tau < \infty) \leq 1$ *if* $\nu(\Lambda) = 0$, *nothing to prove. Assume* $\neq (\Lambda) > 0$. *Note,* $\forall y \in \Lambda$,

$$\int_\Lambda g(x,y) d\nu(x) = \sum_{x \in \Lambda} \nu(x) \sum_{n=0}^{\infty} \mathbf{P}^x(X_n = y) = G(\rho, y)$$

so $\int_\Lambda K(x,y)d\nu(x) = 1$. *Hence* $I_k\left(\frac{\nu}{\nu(\Lambda)}\right) = \nu(\Lambda)^{-1}$ *and* $cap_K(\lambda) \geq \nu(\Lambda)$

For the lower bound, use the second moment method. Given a probability measure μ *on* Λ, *let*

$$z = \int_\Lambda G(\rho,y)^{-1} \underbrace{\sum_{n=0}^{\infty} 1_{X_n=y}}_{\mathbf{E}^\rho()=G(\rho,y)} d\mu(y)$$

By Fubini, $\mathbf{E}^{rho} z = 1$

$$\mathbf{E}^{rho} z^2 = \mathbf{E}^\rho \int_\Lambda \int_\Lambda G(\rho,z)^{-1} G(\rho,z)^{-1} \sum_{m,n>0} 1_{X_m=z,\, X_n=y} d\mu(y) d\mu(z) \leq$$

$$\leq \mathbf{E}^\rho 2 \int_\Lambda \int_\Lambda G(\rho,y)^{-1} G(\rho,z)^{-1} \sum_{0\leq m\leq n} 1_{X_m=z,\, X_n=y} d\mu(y) d\mu(z)$$

For each m, $\mathbf{E}^{rho} \sum_{n\geq m} 1_{X_m=z,\, X_n=y} = \mathbf{P}^{rho}(X_m = z) G(z,y)$ *by the markov property. Taking the sum over* m

$$\mathbf{E}^\rho z^2 \leq 2 \int_\Lambda \int_\Lambda \frac{G(z,y)}{G(\rho,y)} d\mu(z) d\mu(y) = 2 I_K(\mu)$$

By Cauchy-Schwartz:

$$\mathbf{P}^{rho}(\exists n, X_n \in \Lambda) \geq \mathbf{P}^{rho}(z > 0) \geq \frac{(\mathbf{E}^\rho z)^2}{\mathbf{E}^\rho z^2} \geq \frac{1}{2 I_K(\mu)} \Rightarrow \mathbf{P}(\exists n;\ X_n \in \Lambda) \geq \frac{1}{2} cap_K(\Lambda)$$

Proving the first part of the theorem.
For the second part, note that by transience

$$\{X_n \in \Lambda \ i.o\} = stackrelmod\, 0 = \cap_{\Lambda_0 \ finite} \{visits \Lambda \setminus \Lambda_0\}$$

Hence if Λ_n *is a sequence of finite sets increasing to* Λ, *the by monotonicity*

$$\mathbf{P}(visits\ \Lambda\ i.o) = \lim_{n\to\infty} \mathbf{P}^\rho(visits \Lambda \setminus \Lambda_0)$$

and

$$cap_K^{(\infty)}(\Lambda) = \lim_{n\to\infty} cap_K(\Lambda \setminus \Lambda_0)$$

Corollary 16 *In* $\mathbb{Z}^d$, $d \geq 3$, *for any* $A \subseteq \mathbb{Z}^d$, *we deduce*

$$\frac{1}{2} cap_K^{(\infty)}(A) \leq \underbrace{\mathbf{P}(A \ is \ hit \ i.o)}_{\in\{0,1\}\ by\ Hewitt-Savage\ 0-1\ law} \leq cap_K^{(\infty)}(A)$$

So A *is hit i.o a.s if and only if* $cap_K^{(\infty)}(A) > 0$, *now note that this remains true if we replace* K *by* $F(x,y) = \frac{|y|^{2-d}}{1+|x-y|^{2-d}}$ *since by the asyptotics of* $G(x)$ *we have* $c \leq \frac{F(x,y)}{K(x,y)} \leq c'$ *for some* $c', c > 0$.

Remark 17 *In a different application, the B-P-P theorem show a theorem of Lyons that the chance that a path from root to leaves survive under a general Perculation (keep each eage with probability p_e independet) Process is given up to $\frac{1}{2}$ by a certain capacity on leaves, by looking at the Markov chain of surviving leaves from left to right.*

Remark 18

1. *The theorem still applies if the chainis not transient but $G(x,y) < \infty$ $\forall x, y \in \Lambda$*
2. *You can replace K by its symmetrized version $\frac{1}{2}(K(x,y) + K(y,x))$ in the theorem, since it doesn't affect capacity.*

Intersections of Random Walks
Define $S[n,m] := \{S_j;\ j \in [n,m]\}$

Theorem 19 *$\exists c_1, c_2$ and φ such that $\forall n \geq 2$*

$$c_1\varphi(n) \leq \mathbf{P}(S[0,n] \cap S[2n,3n] \neq \emptyset) \leq \mathbf{P}(S[0,n] \cap S[2n,\infty) \neq \emptyset) \leq c_2\varphi(n)$$

$$\varphi(n) = \begin{cases} 1 & d \leq 3 \\ (\log n)^{-1} & d = 4 \\ n^{-\frac{d-4}{2}} & d \geq 5 \end{cases}$$

Proof *Trivial uper bound for $d \leq 3$ and the lower bound for $d \leq 2$ followes from lower bound for $d = 3$, therefore we can assume $d \geq 3$. Proof is by second moment method, with additional trick for $d = 4$. Denote*

$$J_n = \sum_{j=0}^{n} \sum_{k=2n}^{3n} 1_{S_j = S_k},\ K_n = \sum_{j=0}^{n} \sum_{k=2n}^{\infty} 1_{S_j = S_k}$$

By the Markov property and LCLT we get, (when $k - j$ is even)

$$\mathbf{P}(S_j = S_k) = \mathbf{P}(S_{k-j} = 0) \approx \frac{c}{n^{d/2}}$$

So

$$c_1 n^{\frac{4-d}{2}} \leq \mathbf{E}J_n \leq c_2 n^{\frac{4-d}{2}}$$

we can even get (with a bit more calculation)

$$\mathbf{E}J_n^2 \leq \begin{cases} cn & d = 3 \\ c\log n & d = 4 \\ cn^{\frac{d-2}{2}} & d \geq 5 \end{cases}$$

And we get the lower bound for all d since

$$\mathbf{P}(S[0,n] \cap S[2n,3n] \neq \emptyset) \geq \mathbf{P}(J_n > 0) \geq \frac{(\mathbf{E}(J_n))^2}{\mathbf{E}J_n^2}$$

We get the upper bpund for all $d \neq 4$ since

$$\mathbf{P}(S[0,n] \cap S[2n,\infty) \neq \emptyset) = \mathbf{P}(K_n \geq 1) \leq \mathbf{E}K_n$$

For the upper bound in $d = 4$ need to show

$$\mathbf{E}(K_n |\ K_n \geq 1) \geq c \log n$$

this is the expected number of pairs of times of intesec for s SRW starting in origin, this is like what happens after "'first"' intersection of walks.
Note $c = \mathbf{E}K_n = \mathbf{P}(K_n \geq 1)\mathbf{E}(K_n |\ K_n \geq 1)$

Remark 20 *By taking $n \to \infty$ we get that for BM in $\mathbb{R}^d$*

$$\mathbf{P}(B[0,1] \cap B[2,3] \neq \emptyset) \begin{cases} > 0 & d = 1,2,3 \\ = 0 & d \geq 4 \end{cases}$$

From the other side, it is intresting to consider

$$q(n) = \mathbf{P}(S^1[0,n] \cap S^2(0,n] \neq \emptyset)$$

for $d = 2,3,4$ where S^1, S^2 are independet SRW. Defining

$$Y_n = \mathbf{P}(S^1[0,n] \cap S^2(0,n] \neq \emptyset | S^1[0,n])$$

We have $\mathbf{E}Y_n = q_n$, and it is possible to calculate

$$\mathbf{E}Y_n^2 = \mathbf{P}(S^1[0,n] \cap (S^2(0,n] \cup S^3(0,n]) \neq \emptyset) \underset{\text{up to a constant}}{\approx} \begin{cases} \frac{1}{\log n} & d = 4 \\ n^{\frac{d-4}{2}} & 1,2,3 \end{cases}$$

Since $0 \leq Y_n \leq 1$ we have $\mathbf{E}Y_n^2 \leq \mathbf{E}Y_n \leq \sqrt{\mathbf{E}Y_n^2}$
Finally one can show that in $d = 4$ the upper inequality us sharp $q(n) \approx \frac{1}{\log n}$ in $d = 4$
Known: $q(n) \underset{const}{\approx} n^{-f_d}$. *where $f_1 = 1$, $f_2 = \frac{5}{8}$, for $d = 3$ it is an open question, by simulation it is known that $f_3 \approx 0.29$*

References

[1] G. Lawler. Intersection of Random Walks Birkha"user Boston; 1 edition (August 28, 1996)

[2] G. Lawler, V. Limic. Random Walk: a Modern Introduction Cambridge Studies in Advanced Mathematics (No. 123) June 2010

[3] I. Benjamini, R. Pemantle, Y. Peres. Martin Capacity for Markov Chains Ann. Probab. Volume 23, Number 3 (1995), 1332-1346.