More on fluctuations in first passage percolation

Detour to influences in Boolean RCNs.
and Talagrand's inequality (following Benjamini-Kalai-Schramm 2003)

**Hypercube:** $E_0,1^n$
Equipped with the uniform measure $\mu$.
Random variables: $\mathcal{F}: E_0,1^n \rightarrow \mathbb{R}$.

**Influence:** For $1 \leq j \leq n$, let $\xi_j: E_0,1^n \rightarrow \mathbb{R}$

$$(\xi_j \mathcal{F})(x) := \mathcal{F}([x_1, \ldots, x_{j-1}, 1-x_j, x_{j+1}, \ldots, x_n])$$

$\mathcal{F}$ flip $j$th coordinate operation

$$(\xi_j \mathcal{F})(x) := \frac{\mathcal{F}(x) - \mathcal{F}(\xi_j(x))}{2}$$

The function measures the change made by the flip.

**Norms:** $\| \mathcal{F} \|_p := (\mathbb{E}(|\mathcal{F}|^p))^\frac{1}{p}$

$$\| \mathcal{F} \|_p := \left( \frac{1}{2^n} \sum_{x \in E_0,1^n} \mathcal{F}(x)^p \right)^{\frac{1}{p}}$$

**Theorem (Talagrand 1994):**

$$\text{Var}(\mathcal{F}) \leq C \sum_{j=1}^{n} \frac{\| \xi_j \mathcal{F} \|_2^2}{1 + \log \left( \frac{\| \xi_j \mathcal{F} \|_2}{\| \mathcal{F} \|_2} \right)}$$

Without the denominator, this is in the same vein as the Efron-Stein inequality.

Get a significant improvement

if $\| \xi_j \mathcal{F} \|_2 > \| \mathcal{F} \|_2$

This is the case, in our application

when $\mathcal{F}$ is only rarely non-zero.

To show a proof, following...
We show a proof following Benjamini–Kalai–Schramm (2003), 1970 & 1975.
The proof relies on the Bonami–Beckner inequality:

**Bonami–Beckner**: noise operator $N_\epsilon$ ($0 \leq \epsilon \leq 1$) takes a function $f : \mathbb{Z}_2^n \to \mathbb{R}$ and returns another function which is somewhat averaged.

Let $X_1, \ldots, X_n$ be Bernoulli $(\epsilon)$ independent random variables:

$$X_i \sim \begin{cases} 1 & \epsilon \\ 0 & 1-\epsilon \end{cases}$$

$$N_\epsilon f(x) = \mathbb{E}[f(x + X \mod 2)]$$

Each coordinate is flipped with probability $\epsilon$ and we average $f$ on the resulting random position.

**Theorem (Bonami–Beckner)**: For each $0 < \epsilon < 1$

$$\|N_\epsilon f\|_2 \leq \|f\|_1 + 8\epsilon$$

Hypercontractive inequality

For $f(\epsilon) = (1 - 2\epsilon)^2$,

**Representation in a Fourier–Walsh basis**:

For a subset $S \subseteq \{1, \ldots, n\}$, let

$$X_S : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$$

$$X_S(x) = \sum_{i \in S} x_i$$

This is an orthonormal basis for $L^2$ of functions on the hypercube with the uniform measure.

**Orstein** write $f(x) = \sum \hat{f}(S) x_S(x)$.
Parseval: Write \( P(x) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x) \) w.r.t. the uniform measure in the expansion.

\[ \| P \|_2^2 = \sum_{S \subseteq [n]} |\hat{f}(S)|^2. \]

Noise operator: What is the expansion of \( N_\varepsilon x \)?

Since \( P(x) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x) \),

then \( N_\varepsilon x = \sum_{S \subseteq [n]} \hat{f}(S) \mathbb{E}(x_S(l_\varepsilon l + x) \mod 2) \chi_S(x) \).

\[ = \sum_{S \subseteq [n]} \hat{f}(S) \mathbb{E}
\left[
\left(-\varepsilon \sum_{i \in S} x_i + x \right)
\right] =
\]

\[ = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x) \prod_{j \in S} \mathbb{E}
\left[
-\varepsilon x_j
\right] =
\]

\[ = \sum_{S \subseteq [n]} (1-2\varepsilon)^{|S|} \hat{f}(S) \chi_S(x). \]

Useful notation: For \( -1/p \leq 1 \), write \( T_p \) for the operator taking \( f \) to

\[ T_p (f)(x) = \sum_{S \subseteq [n]} p^{|S|} \hat{f}(S) \chi_S(x) \]

So that \( T_{1-2\varepsilon} = N_\varepsilon \).

In this notation, Bonami-Beckner says that

\[ \| T_p \|_2 \leq \| P \|_2 + p^2. \]

Proof of Talagrand's inequality:
Proof of Talagrand's inequality:

Write \( f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x_S(x) \).

Recall \((p, f)(x) = \frac{f(x) - \langle \phi, f \rangle}{2}\) input bit operation.

What is the Fourier-Walsh expansion of \( p_j f \)?

Simple to see that \((p, f)(x) = \begin{cases} \hat{f}(S)(x) & \text{j.e.} S \\ -\hat{f}(S)(x) & \text{else} \end{cases} \)

\( \Rightarrow p_j(f)(x) = \sum_{S \subseteq [n]} \hat{f}(S) x_S(x) \).

How to express the variance?

\[
\text{Var}(f) = \mathbb{E}(f - \mathbb{E}(f))^2 = \sum_{\phi \neq S \subseteq [n]} \mathbb{E} \hat{f}(S)^2.
\]

How to use the Bonami-Beckner ineq.?

\[
\text{Var}(f) = \sum_{\phi \neq S \subseteq [n]} \mathbb{E} \hat{f}(S)^2 \leq \sum_{j=1}^n \sum_{S \subseteq [n]} \mathbb{E} \hat{f}_j(S)^2.
\]

\[
\leq 3 \sum_{j=1}^n \left\| \mathbb{E} (p_j f) \right\|^2 dp_j
\]

Fourier-Walsh expansion = \( \sum_{S \subseteq [n]} \hat{f}(S) x_S(x) \)

\( \Rightarrow \left\| \mathbb{E} (p_j f) \right\|^2 = \sum_{S \subseteq [n]} \hat{f}_j(S)^2 \).

Main inequality:

Bonami-Beckner \( \Rightarrow \text{Var}(f) < a^2 \left\| \mathbb{E} f \right\|^2 \).
Main inequality.

Bonami–Beckner \( \Rightarrow \) \( \text{Var}(K) \leq \sum_{\mathcal{S}} \frac{\epsilon^2}{2} \sum_{j=1}^{n} \|P_{j}\|_2 \frac{2}{1+p^2} dp \).

Algebraic manipulations:

\[
\begin{align*}
\mathbb{E}(g_1^2 + p^2) &= \mathbb{E}[g_1^2 \cdot g_1^2] \cdot \mathbb{E}[g_1^2] \\
&= \|g_1\|_2^2 \cdot \|g_1\|_1 \\
&= \|g_1\|_2^2 (\frac{\|g_1\|_1}{\|g_1\|_2})^{\frac{2(1-p^2)}{1+p^2}}.
\end{align*}
\]

Also note that for \( 0 \leq x \leq 1 \):

\[
\int_0^1 \frac{2(1-p^2)}{1+p^2} dp \leq 1 - \frac{x}{\log(\frac{1}{x})}.
\]

Putting everything together:

\[
\text{Var}(K) \leq \sum_{\mathcal{S}} \frac{\epsilon^2}{2} \sum_{j=1}^{n} \|P_{j}\|_2 \frac{2}{1+p^2} dp \cdot \frac{1 - (\|P_j\|_1 \|P_j\|_2)}{\log(\|P_j\|_1 \|P_j\|_2)}.
\]

One checks

\[
\sum_{j=1}^{n} \frac{\|P_{j}\|_2^2}{1 + \log(\|P_{j}\|_2 \|P_{j}\|_2)} \leq C \sum_{j=1}^{n} \frac{\|P_{j}\|_2^2}{1 + \log(\|P_{j}\|_2 \|P_{j}\|_2)}.
\]

Finishing the proof of Talagrand's ineq.

Application to first passage percolation

Consider \( \text{FPP} \) on \( \mathbb{Z}^d \) in which the edge weights are uniform on \( [0,6]\) for some \( 0 < a < 6 < \infty \).
For some \( 0 < a < b < \infty \).

Assume \( d \geq 2 \).

Consider \( T(0, L e_1) \) \( (e_i = (1, 0, 0, \ldots, 0), \text{first coord.incr.}) \)

which is the shortest passage time from 0 to \( L e_1 \).

**Thm. (Benjamini-Kalai-Schramm 2003):**

\[
\text{Var}(T(0, L e_1)) \leq c \frac{L}{\log L}.
\]

(Best known upper bound)

(Believed in \( d = 2 \) that \( \text{Var} \sim L^{3/2} \)
and even smaller in \( d \geq 3 \)).

Note that the optimal path can use at most \( \frac{b - a}{a} \cdot L \) edges since the straight line path is a candidate.

Thus \( T(0, L e_1) \) is a fcn. of finitely many Boolean random variables. So Talagrand's ineq. applies.

For brevity, write \( f = T(0, L e_1) \)

a fcn. of \( (\eta e) \), where \( \eta e \in \{-a, b\} \) is the weight of the edge \( e \).

Note \( \eta e f = \frac{f - \sigma e f}{2} \), where \( \sigma e f \) is the passage time after flipping the weight of the edge \( e \).

**Note:** if \( \eta e = a \) then \( \sigma e f \geq f \) always.
Note: if $\omega_e = a$ then $\omega_e \not= \omega_f$ always
and $\omega_e \not= \omega_f$ iff all shortest paths
from 0 to $L_e$ pass through $e$.
Thus $1E(\omega_e \omega_f) = \frac{1}{2} 1E(\omega_e \omega_f | \omega_e = a) =$

$$= P(\text{all optimal paths } | \omega_e = a) \cdot c(a, b).$$

To improve upon Efron-Stein, need
that $\frac{11\omega_e \omega_f^{1/2}}{11\omega_e^{1/2}}$ is large.

But $\frac{11\omega_e \omega_f^{1/2}}{11\omega_e^{1/2}} = \frac{c(a, b)}{\sqrt{P(\text{all optimal paths } | \omega_e = a)}}$

This probability is expected to be small for edges far from 0 and $L_e$; probably as small as $\frac{1}{L}$,
and this would imply the BKS upper bound on the variance
by Talagrand’s Ineq.

Open question: Prove such an upper bound
on the probability (BKS midpoint problem).
Known: The above prob. tends to 0 with $L$
for $e$ far from 0 and $L_e$.

(Ahlberg-Hoffman 2016)

Idea to bypass the open question:
Idea to bypass the open question:

Instead of \( T(0, \mathbb{L}_1) \) consider an averaged version:

\[
\mathbb{S}(0, \mathbb{L}_1) := \frac{1}{|B(0, m)|} \sum_{Z \in B(0, m)} T(Z, Z + \mathbb{L}_1)
\]

where \( B(0, m) := \{ Z \in \mathbb{Z}^d : ||Z|| \leq m \} \)

and choosing \( m = L^{1/4} \).

Certainly, \( \mathbb{E}\mathbb{S}(0, \mathbb{L}_1) = \mathbb{E}T(0, \mathbb{L}_1) \)

by translation invariance.

Additionally, \( |\mathbb{S}(0, \mathbb{L}_1) - T(0, \mathbb{L}_1)| \leq C(a, b) \cdot m \)

\[\Rightarrow \text{Var}(T) = \|T - \mathbb{E}T\|_2^2 \leq \|S - \mathbb{E}\mathbb{S}\|_2^2 = \frac{2b^2}{C(a, b) m^2} \leq \text{Var}(\mathbb{S})\]

\[\Rightarrow \text{It suffices to use Talagrand's ineq. for } \mathbb{S}. \text{ The advantage is that for } \mathbb{S}, \text{ due to the averaging, one can show that } \|T - \mathbb{E}T\|_2 \text{ is a power of } L^{1/4}\]

without much difficulty (see 50 years of FPP survey).

Final remarks: The upper bound \( \text{Var}(T) \leq \frac{8L}{\log L} \)

is now known for most weight dist.

Talagrand's ineq. is sometimes replaced by Falik-Smorodnitsky's ineq.

Damron-Hanson-Sassoe 2015
Detour to total variation distances
and a lower bound on the variance

Best-known lower bound on the variance,
Thm. (Newman-Piza 1995): In $d=2$,

$$\text{Var}(T(0,1e_j)) \geq c \log_2.$$ 

Open whether $\text{Var} \xrightarrow{L^2} \infty$ in $d \geq 3$.
(Open even in physics literature).

**Total variation distance:**

$\mathcal{N}, \mathcal{V}$ are prob. measures on some measure space.

$$d_{TV}(\mathcal{N}, \mathcal{V}) = \sup_{A \text{ event}} |\mathcal{N}(A) - \mathcal{V}(A)|$$

For every $A$, $\mathcal{V}(A) - d_{TV}(\mathcal{N}, \mathcal{V}) \leq \mathcal{N}(A) + d_{TV}(\mathcal{N}, \mathcal{V})$

When $\mathcal{N}, \mathcal{V}$ have densities wrt. to a third measure $\lambda$ (always okay with, say $\lambda = \mathcal{L}^d$)
(Usually, $\lambda = \text{Lebesgue or counting}$)

Write $\mathcal{N} = f \, d\lambda$, $\mathcal{V} = g \, d\lambda$.

**Claim:**

$$d_{TV}(\mathcal{N}, \mathcal{V}) = \frac{1}{2} \int |f - g| \, d\lambda = S(f - g) \mathbf{1}_{f > g} \, d\lambda = S(g - f) \mathbf{1}_{g > f} \, d\lambda.$$

**Proof:**

First, $\frac{1}{2} \int |f - g| \, d\lambda = \int \left[ \frac{1}{2} (S(f - g) \mathbf{1}_{f > g} + S(f - g) \mathbf{1}_{g > f}) \right] \, d\lambda = \int \left[ S(f - g) \mathbf{1}_{f > g} + S(g - f) \mathbf{1}_{g > f} \right] \, d\lambda$.
\[
= \frac{1}{2} \left[ S(\theta \cdot g) \mathbf{1}_{F \geq g} \, d\lambda + S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \, d\lambda \right] \\
= 0
\]

\[= S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \, d\lambda.\]

Now, for any event \(A\),
\[
N(A) = S_K \mathbf{1}_{A} \leq S_g \mathbf{1}_{A} + S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \mathbf{1}_{A} \leq V(A) + S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \mathbf{1}_{A}
\]
\[
\Rightarrow \quad d_{TV}(N(A)) \leq S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \mathbf{1}_{A}.
\]

For other direction, if \(A = \{F > g\}\) then
\[
N(A) = S_F \mathbf{1}_{A} = S_g \mathbf{1}_{A} + S(\theta - \mathbf{1}) \mathbf{1}_{F \geq g} \mathbf{1}_{A}
\]
\[
= V(A)
\]

It is not always easy to give an upper bound to \(d_{TV}(N, D)\). We discuss now one of the simplest cases.

**Example:** \(X = (X_1, \ldots, X_n) \overset{iid}{\sim} X; \sim \text{Norm}(0, 1)\)

and \(Y = (Y_1, \ldots, Y_n) \overset{iid}{\sim} Y\) indp.

where \(Y_i \sim N(\varepsilon_i, 1)\).

How big is \(d_{TV}(L(X), L(Y))\)?

Claim: \(d_{TV}(L(X), L(Y)) = \Phi\left(\frac{1}{\sqrt{2\pi} \varepsilon_2} |X - \varepsilon_1| \leq \frac{1}{2}\right)\)
Claim: $d_{TV}(\mathcal{L}(X), \mathcal{L}(Y)) = P\left( \frac{1}{2\|\varepsilon\|_2} |X, \varepsilon| < \frac{1}{2\|\varepsilon\|_2} \right)$

$\approx \frac{e^{\|\varepsilon\|_2}}{\|\varepsilon\|_2} e^{-\frac{1}{8}\|\varepsilon\|_2^2} \quad \|\varepsilon\|_2 << 1$

$\approx 1 - \frac{e}{\|\varepsilon\|_2} e^{-\frac{1}{8}\|\varepsilon\|_2^2} \quad \|\varepsilon\|_2 >> 1$

Proof by picture: densities for $X$ and $Y$

Application to directed last passage percolation with Gaussian weights

$\tau(0, n) =$ maximal sum of weights over all right-up paths from $(0,0)$ to $(n,n)$.

Here we take the weights to be IID $\mathcal{N}(0,1)$.

Claim: $\text{Var}(\tau(0,n)) \geq c \log n$.

Proof: Let $(w_e)$ be the edge weights.

Define new weights $\tilde{w}_e = w_e + \tilde{d}_e$,

where $(\tilde{d}_e)$ are deterministic and depending only on $|e| :=$ distance of $e$ from $(0,0)$.

Define $\tau'(0,n)$ to be like $\tau(0,n)$ but with the weights $(\tilde{w}_e)$.

Clearly, $\tau'(0,n) \geq \tau(0,n) + \sum_{|e| = 1} \tilde{d}_e$

Since the old geodesic is a candidate for the optimal path (and actually it equals the)

maximal path.)
Since the optimal path (and actually it equals the optimal path)

Now consider the event

\[ A = \{ \frac{1}{n} \left| T(0,n) - P(T(0,n)) \right| \leq \frac{1}{n} \} \]

Then \[ \Pr(\eta \in A) - \Pr(\tilde{\eta} \in A) \leq d_{TV}(\lambda(\eta), \lambda(\tilde{\eta})) \]

If we take \( t \) to be, say, two standard deviations of \( T(0,n) \) then by Chebyshev's inequality, \( \Pr(\eta \in A) \geq \frac{3}{4} \).

Thus, if \( d_{TV}(\lambda(\eta), \lambda(\tilde{\eta})) \leq \frac{1}{2} \), say, then \( A \) is likely also for \( \tilde{\eta} \).

\[
\begin{align*}
\text{Hence } \quad \sum_{|e| = 1}^{2n} \delta_{e} e & \leq 2t = \text{Four standard deviations of } T(0,n) \\
\end{align*}
\]

Conclude: \( \text{std}(T(0,n)) \geq C \sum_{|e| = 1}^{2n} \delta_{e} \)

Whenever \( d_{TV}(\lambda(\eta), \lambda(\tilde{\eta})) \leq \frac{1}{2} \).

Choice of \( \delta_{e} \): Choose \( \delta_{e} = \frac{C}{|e| \sqrt{\log \eta}} \).

Then \( d_{TV}(\lambda(\eta), \lambda(\tilde{\eta})) \leq \| \delta_{e} \|_{2} = \sum_{|e| = 1}^{2n} \delta_{e}^{2} |e| \leq \frac{1}{2} \).

\[
\Rightarrow \text{std}(T(0,n)) \geq C \sum_{|e| = 1}^{2n} \delta_{e} = C \sqrt{\log \eta}.
\]

Mermin-Wagner style arguments