Localization of random band matrices when $W \ll N^{1/4}$

Wednesday, 22 March 2023  9:36

Random band matrix model:

$$H = \begin{pmatrix}
V_1 - T_1^* & -T_1 & 0 & \\
-T_1 & V_2 - T_2^* & -T_2 & 0 \\
0 & -T_2 & V_3 - T_3^* & -T_3 \\
0 & 0 & 0 & V_n - T_n^*
\end{pmatrix}$$

- Hermitian
- $W \times W$ block matrix
- Dimension $N=nW$
- Bandwidth $D$
- Independent Gaussian entries up to Hermitian constraint
- Wegner orbital model

$(V_j)$ independent GUE$(W)$ matrices

$(T_j)$ independent Ginibre$(W)$ matrices

Density $p(H) = \frac{1}{Z_{n,W}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \|V_j\|_F^2 + \sum_{j=1}^{n} \|T_j\|_F^2\right)$

With respect to Lebesgue measure $dV_n dT_n dT_{n-1} \cdots dT_1$ on complex $W \times W$ Hermitian matrices and complex $W \times W$ matrices.

Conjecture (Casati-Molinari-Zaccher 1990): $H$ exhibits localization when $W \ll N^{1/2}$

Our result (Cipolloni-Feffer-Schenker-Shaprio 2022): Localization holds when $W \ll N^{1/4}$.

Previous and later results:

Localization: Schenker (2009) $W \ll N^{1/8}$

P.-Schenker-Sodin-Shemie (2017) $W \ll N^{1/2}$

In parallel to us, Smart-Chen (2022) $W \ll N^{1/9}$

Afterward, Goldstein (2022) $W \ll N^{1/2}$

M.O.T. Scherbina (series of works) Crossover at $N^{1/2}$ for related questions

Poisson statistics proved only when $W$ fixed, $N \to \infty$

(Dude-Hislop 2020)

Delocalization results by Bao, Bourgade, Erdős, Knowles, Yang, You, Yin (2011-2017)

State-of-the-art by Bourgade, Yang, Yao, Yin $W \gg N^{3/4}$

Overview of proof:

For energy level $E$, let $G_E = (H - E I)^{-1}$.
For energy level \( E \in \mathbb{R} \), let \( G_z := (H-zI)^{-1} \) be the resolvent, or Green's function, of \( H \). Think of \( G_z \) as a \( W \times W \) block matrix and let \( G_z(x,y) \) be its \((x,y)\) block \((1 \leq x,y \leq W)\).

**Our result:** \( \forall z \in \mathbb{R} \), \( \forall \delta > 0 \) s.t. \( \forall x,y \)
\[
E(\| G_z(x,y) \|_\infty^6) \leq W^2 e^{\exp(-\frac{1}{\delta} W^\frac{1}{3})}
\]
E.g., operator norm

This is known to imply localization when \( n < \ll W^\frac{3}{2} \) \((\Leftrightarrow n \ll W^2)\)

For simplicity, fix \( z = 0 \).
Write \( G_z \) for \( G_{z = 0}(z,0) \) and focus on result for \( G_n \).

**Basic approach (Schenker 2009, suggestion of Aizenman):**

1. **a-priori bound:**
   \[
P(\| G_n \|_\infty > t) \leq \frac{\text{poly}(W,n)}{t^2}, \quad t > 1.
   \]

2. **Explicit Formula:**
   \[
   G_n = T_1^{-1} T_1^* T_2^* T_2^{-1} \cdots T_{n-1} T_{n-1}^* T_n^{-1}
   \]
   Where
   \[
   T_i := V_i, \quad T_j := V_j - T_{j-1} T_{j-1}^* T_{j-1}^{-1} \quad \text{for } 2 \leq j \leq n. \tag{A}
   \]

3. The formula suggests that \( \| G_n \|_\infty \) is like a sum of independent contributions and hence should have wide spread. However, the a-priori bound shows \( \| G_n \|_\infty \) is unlikely to be large. Hence \( \| G_n \|_\infty \) should typically be small.

We will formalize this to prove
\[
P(\| G_n \|_\infty \geq e^{-c_{\frac{n}{W^3}}}) \leq C e^{-c_{\frac{n}{W^3}}}
\]

Together with the a-priori bound, this will imply that for \( s \in (0,1) \),
\[
E(\| G_n \|_S) \leq W^2 e^{-c_{\frac{n}{W^3}}}
\]

(1) and (2) are standard. We explain here our formalization of (3).

Details from proof:
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1) Change of variables: The \((T_i)\) are Hermitian by \((\mathcal{O})\). We change variables from \((V, T)\) to \((\Gamma, \Upsilon)\). By \((\mathcal{O})\) the Jacobian of this transformation is \(1\). Hence the density of \((\Gamma, \Upsilon)\) is

\[ p(\Gamma, \Upsilon) = \frac{1}{Z_{\text{norm}}} \exp(-\mathcal{W}(\sum_{i} |T_i|^2 + \sum_{i} \Upsilon_i + \sum_{i} |T_i|^2 + \sum_{i} \Upsilon_i)) \]

where we put \(T_{i-1} = 0\) for \(i = 1\).

2) Inducing fluctuations (an "adaptive Mermin-Watson" transformation).

We seek to perturb \((\Gamma, \Upsilon)\) in a way that doesn't change the density \(p\) too much but does change our target observable \(\langle \text{target observable} \rangle\) significantly.

Lemma (with origins in Friiser 1987, Richenderfer 2007, Mitloehner 2015):
Let \(X\) be a random variable in \(\mathbb{R}^m\) with density \(p\). Let \(s^+, s^- : \mathbb{R}^m \to \mathbb{R}^m\) be absolutely continuous bijections. Then for event \(E\)

\[ \sqrt{p(X \in s^+(E)) p(X \in s^-(E))} \geq \lambda(E, s^+, s^-) p(X \in E) \]

where

\[ \lambda(E, s^+, s^-) = \inf_{x \in \mathbb{R}^m} \left[ \frac{\sqrt{p(s^+(x)) p(s^-(x))}}{p(x)} \right] \]

\[ \left( s^+(E) \right) \]

with \(J^\pm : \mathbb{R}^m \to \mathbb{R}\) are the Jacobians:

\[ J^\pm(x) := \det (\frac{\partial s^\pm}{\partial x_i}(x)) \]

(So that \(p(X \in s^\pm(E)) = \int_{s^\pm(E)} p(x) dx = \int_{s^\pm(E)} \rho(s^\pm(x)) J^\pm(x) dx)\)

**Proof:** Set \(I := \int_{E} \sqrt{p(s^+(x)) p(s^-(x)) J^+(x) J^-(x)} dx\).

[Further details or equations may follow here...]
Then, on the one hand,
\[ I \geq \lambda(E, s^+, s^-) \int_E \rho(x) dx = \lambda(E, s^+, s^-) \int \rho(x) dx. \]
On the other hand, by \( s = s^0 \)
\[ I \leq \left( \int_E \rho(s^+(x)) s^+(x) dx \right)^{1/2} \left( \int_E \rho(s^-(x)) s^-(x) dx \right)^{1/2} \]
\[ = \sqrt{\rho(x \in s^+(E)) \rho(x \in s^-(E))}. \]

3) How to use the above lemma:
we define the mappings \( s^{T \pm} = (t^T, t^{-T}) \)
with \( T^T = e^T F_T^T T \)
with a small \( \delta > 0 \)
and \( F_T = F_T((1/\delta)_T, \gamma) T_{\gamma, \delta}, T_{\gamma, \delta}, T_{\gamma, \delta}^{-T}, T_{\gamma, \delta} \in [0, 1] \)
is a smooth function which typically equals \( T \) but
smoothly transitions to \( 0 \) if any of its arguments
is abnormally large \( \|T_{\gamma, \delta}\| > 1 \) (typically, \( \|T_{\gamma, \delta}\| \not\leq 1 \) but this relaxed bound suffices for us)
These are bijections when \( \delta \gamma \ll 1 \).
It is proved that
\[ \lambda(A_{s^{T \pm}}, s^+, s^-) \geq e^{-c_0 \delta^2 n \delta^{-3}}. \]

Let
\[ E = \{ \|G_n\| \geq e^{-6 \delta n \delta^{-3}} \} \cap \{ \gamma \leq F_T \geq \frac{n}{2} \} \]
By lemma,
\[ \sqrt{\rho((t^T, t) \in s^+(E))} \geq \sqrt{\rho((t^T, t) \in s^+(E)) \rho((t^T, t) \in s^-(E))} \geq \rho(E) \]
\[ \geq e^{-c_0 \delta^2 n \delta^{-3}} \]
\[ \rho(E) \geq \rho(\|G_n\| \geq e^{-6 \delta n \delta^{-3}}) - \rho(\text{at least half of the } F_T \text{ are not } T) \]
\[ \leq e^{-c n} \]

\( \lambda \) uses result or
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Aizenman-P. Sniady-Sohv (2017)
to control norm of \( T_{\gamma, \delta}^{-1} \)
(inverse of \( G_{T_{\gamma, \delta}} + \text{auxiliary hermitian} \) is under control)

\[ \rho((t^T, t) \in s^+(E)) \leq \rho(\|G_n\| \geq e^{-6 \delta n \delta^{-3}}) \leq \rho(\|G_n\| \geq e^{-6 \delta n \delta^{-3}}) \]
\[ \leq \rho(G_n) e^{-\frac{1}{8} \delta n + \frac{c_0}{8} \delta^{-3}}. \]

Thus,
\[ \rho(\|G_n\| \geq e^{-6 \delta n \delta^{-3}}) \leq \rho(G_n) e^{-\frac{1}{8} \delta n + \frac{c_0}{8} \delta^{-3}} + e^{-c n} \]
Thus, $P(\|G_n\| \geq e^{\frac{Cn}{n^2 + 2n}}) \leq \text{poly}(\text{tr}^*) e^{-\frac{Cn}{n^2 + 2n}} + e^{-cn}\$.

Choose $\delta = \frac{1}{8c_0 w^3}$ to get $P(\|G_n\| \geq e^{\frac{Cn}{n^2 + 2n}}) \leq \text{poly}(\text{tr}^*) e^{-\frac{Cn}{n^2 + 2n}} + e^{-cn}\$ as we wanted to prove.

4) The bottleneck for $(**)$:

Examine $\inf\limits_{(T, T) \in E} \sqrt{P(S + (T, T)) P(S + (T, T))}$.

To estimate it, use that $A, B, C$

$\frac{1}{2} (\|A + B + C\|^2_{HS} + \|A - B + C\|^2_{HS}) - 2\|A\|^2_{HS} = \|B\|^2_{HS} + 2\text{Re} (\text{tr}(C^* A))$

The contributing term in the density $\rho$ is

$\|T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp \|^2_{HS}\$.

We have

$\begin{align*}
\hat{T}_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp &= T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp + 2\delta F_{u-1} \\
&= T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp + T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp (e^{-2\delta F_{u-1}})
\end{align*}$

$A = V_u$.

So that $B \approx \delta F_{u-1} T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp = \delta F_{u-1} (V_u - V_0)$.

$C \approx \delta^2 F_{u-1}^2 T_{u-1}^{-1} T_{u-1}^* T_{u-1}^\perp = \delta^2 F_{u-1} (V_u - V_0)$.

and we have $\|B\|^2_{HS} \leq \delta^2 w^2 \quad a \text{ This is the main contributing factor}$

$\text{Re} (\text{tr}(C^* A)) \leq \delta^2 w^2 \quad a \text{ This estimate can be reduced with tighter control on } \|C\|_{\text{HS}}$

as $F_{u-1} (\|V_u\|_{\text{HS}} + \|V_0\|_{\text{HS}}) \leq w$.

Recalling the extra factor of $w$ multiplying the norms in the density $\rho$, we get a factor of $\delta^3 w^3$ in $(**)$.