Quantitative estimates for the effect of disorder on low-dimensional lattice systems

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Lattice systems with compact state space

- We discuss statistical physics systems on $\mathbb{Z}^d$, aiming to develop a quantitative understanding of the effect of adding disorder to them.

- We start with the case of a compact state space.

- Setup: (1) Compact metric space $S$ equipped with a Borel measure $\kappa$.

  (2) Translation-invariant finite range and finite energy Hamiltonian $H$.

- As usual, for a finite domain $\Lambda \subset \mathbb{Z}^d$, at temperature $T$ and with boundary conditions $\tau: \mathbb{Z}^d \to S$, configurations $\sigma: \mathbb{Z}^d \to S$ coinciding with $\tau$ outside $\Lambda$ are sampled from the probability measure with density

$$\frac{1}{Z_{T, \Lambda, \tau}} \exp \left( -\frac{1}{T} H_\Lambda(\sigma) \right)$$

with respect to the measure $\prod_v d\kappa(\sigma_v)$, where $Z_{T, \Lambda, \tau}$ is the partition function and $H_\Lambda$ contains the terms in the Hamiltonian depending on the spins in $\Lambda$.

- Periodic boundary conditions and the zero-temperature limit are also allowed.

- Examples:
  - Ising model: $S = \{-1, 1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$
  - Potts model: $S = \{1, 2, ..., q\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} 1_{\sigma_u = \sigma_v}$
  - Spin $O(n)$ model with $n \geq 2$: $S = S^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2$
Phase transitions in pure systems I

- Ising model: $S = \{-1, 1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u\sim v} \sigma_u \sigma_v$.
- The Ising model undergoes a phase transition in dimensions $d \geq 2$ as the temperature is lowered, from a disordered to an ordered state.
- Similar behavior for the q-state Potts model ($S = \{1, 2, \ldots, q\}$, $H(\sigma) = -\sum_{u\sim v} 1_{\sigma_u=\sigma_v}$).

Simulation from Spinka–Peled 2019

Simulation from Beffara

d=2 Potts model with q=4 at criticality. Simulation by Beffara.

Simulation from Spinka–Peled 2019
Phase transitions in pure systems II

- Spin $O(n)$ model with $n \geq 2$: $S = S^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2$

- **Mermin–Wagner theorem**: The spin $O(n)$ model does not exhibit an ordered phase in two dimensions (even at low temperature).

- **Fröhlich–Simon–Spencer theorem**: A low-temperature ordered phase exists in dimensions $d \geq 3$.

- In two dimensions:
  - $n=2$ (XY model): Berezinskii–Kosterlitz–Thouless transition (proof by Fröhlich–Spencer) from exponential to power-law decay of correlations as temperature is lowered.
  - $n=3$ (Heisenberg model): Polyakov conjecture – exponential decay at all temperatures.

XY model simulation from Spinka–Peled 2019

Heisenberg model simulation from Spinka–Peled 2019
Disordered lattice systems

• Noised observables: Let $f: S^d \rightarrow \mathbb{R}^m$, for some $m \geq 1$, be a bounded measurable function depending on the spins in a finite neighborhood of the origin.

Disorder: Let $(\eta_v)_{v \in \mathbb{Z}^d}$ be independent standard $m$-dimensional Gaussian vectors.

Disordered Hamiltonian: $H^\eta(\sigma) = H(\sigma) - \lambda \sum_v \eta_v \cdot f(\mathcal{T}_v(\sigma))$
where $\mathcal{T}_v(\sigma)$ is the configuration $\sigma$ translated by $v$.

• Examples:

Random-field Ising model: $m = 1$ and $f(\sigma) = \sigma_0$. Thus

$$H^\eta(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$$

• Edwards-Anderson spin glasses: $S = \{-1, 1\}, \mu = \text{counting}, f(\sigma) = (\sigma_{e_j} \sigma_0)^d_{j=1}$.

$$H^\eta(\sigma) = -\lambda \sum_{u \sim v} \eta_{u,v} \sigma_u \sigma_v$$

• Random-field $q$-state Potts model: $m = q$ and $f(\sigma) = (1_{\sigma_0=1}, \ldots, 1_{\sigma_0=q})$. Thus

$$H^\eta(\sigma) = -\sum_{u \sim v} 1_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_{k=1}^q \eta_{v,k} 1_{\sigma_k=k}$$

• Random-field spin $O(n)$ model, $n \geq 2$: $m = n$ and $f(\sigma) = \sigma_0$ (with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$),

$$H^\eta(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2 - \lambda \sum_v \eta_v \cdot \sigma_v$$
Imry-Ma phenomenon

- **Imry-Ma (1975)** considered the effects of disorder for the random-field Ising and spin $O(n)$ models, and predicted that in low dimensions, an arbitrarily small disorder strength $\lambda$ causes the models to lose their ordered phase, as follows:
  - The random-field Ising model is disordered at all temperatures for $d \leq 2$.
  - The random-field spin $O(n)$ model is disordered at all temperatures for $d \leq 4$.
- **Aizenman-Wehr (1989)** proved the predictions as part of a general statement.

- **Notation**: Write $\Lambda_L := \{-L, \ldots, L\}^d$. For each disorder $\eta$, write $\langle \cdot \rangle_\mu$ for the thermal expectation according to a Gibbs measure $\mu$ of the $\eta$-disordered system. Write $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation operator over $\eta$.

- **Theorem** (Aizenman-Wehr, special case): For a disordered lattice system with compact state space (as discussed above) in dimensions $d = 1, 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, the limit
  \[
  \lim_{L \to \infty} \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle f(T_v(\sigma)) \rangle_\mu
  \]
  exists and has the same value for all Gibbs measures $\mu$ and almost all $\eta$.
  The same holds in dimensions $1 \leq d \leq 4$ for the spin $O(n)$ models with $n \geq 2$.

- **Our goal**: Develop a quantitative understanding of this phenomenon.
Random-field Ising model

- Random-field Ising model Hamiltonian: \( H^\eta(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v \)

- The disordered model still satisfies the usual **monotonicity (FKG)** properties. In particular, the model has maximal and minimal Gibbs measures \( \mu^{\eta,+} \) and \( \mu^{\eta,-} \), arising in the thermodynamic limit from constant boundary conditions. The Aizenman-Wehr theorem implies that \( \mu^{\eta,+} = \mu^{\eta,-} \) in two dimensions \( \eta \)-almost surely, so that the model has a unique Gibbs measure.

- A natural **quantitative parameter** is \( m_L := \mathbb{E}\left(\langle \sigma_0 \rangle^+_L\right) \) where \( \langle \cdot \rangle^+_L \) denotes the thermal expectation in \( \{-L, ..., L\}^2 \) with +1 boundary conditions.

- A bound of the form \( m_L \leq \exp(-c(\lambda, T)L) \) is relatively simple for large disorder strength \( \lambda \) or high temperatures \( T \), so interested in small \( \lambda \) and low temperature.

- **Results:** In dimension \( d = 2 \): \( m_L \leq \frac{C(\lambda)}{\sqrt{\log \log L}} \) (Chatterjee 2017), \( m_L \leq \frac{C(\lambda)}{L c(\lambda)} \) (Aizenman-P. 2018) and finally

\[
m_L \leq C(\lambda) \exp\left(-\frac{L}{\ell(\lambda)}\right)
\]

proved at zero temperature by Ding-Xia 2019 and then at positive temperature by Ding-Xia 2019 and Aizenman-Harel-P. 2019.

- Still **open** to determine correlation length \( \ell(\lambda) \) for small \( \lambda \). Proof implies \( \ell(\lambda) \leq e^{e^{1/\lambda^2}} \) (Bar-Nir 2022). Ding-Wirth (2020): Correlation length = \( e^{\Theta(\lambda^{-4/3})} \) in another sense.
Random-field Ising and Potts models

- Dimension $d \geq 3$, weak disorder (small $\lambda$): Imbrie 1985 (zero temperature) and Bricmont-Kupiainen 1988 (all temperatures) established long-range order in the random-field Ising model. A shorter argument was given recently by Ding-Zhuang (2021), also extending the result to the random-field Potts model.

- Ding-Liu-Xia (2022), making use of Ding-Song-Sun (2023), extend the long-range order result to all temperatures lower than the critical temperature of non-disordered Ising model. Ding-Huang-Xia (2023) investigate the critical scaling for the disorder at the critical temperature of the non-disordered Ising model.

- Rigas (2022) extended part of the correlation length result of Ding-Wirth to the random-field Potts model.
Quantitative results

- The other models discussed (Potts, spin-glasses, spin $O(n)$) do not share the monotonicity properties of the random-field Ising model and the proof techniques break down for them. Indeed, even the choice of which quantity to bound is non-obvious since it is unclear which boundary conditions $\tau$ maximize or minimize the average $\langle f(T_v(\sigma))\rangle_{\Lambda_L^2}^{\tau}$ and, indeed, it may be that these boundary conditions depend on the disorder $\eta$ and on $L$ and $v$. We obtain the following results.

- Theorem (Dario-Harel-P 2020+): For each two-dimensional disordered lattice system of the type described above, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E}\left(\sup_{\tau_1, \tau_2: \mathbb{Z}^2 \to S} \left\| \frac{1}{L^2} \sum_{v \in \Lambda_L^2} \langle f(T_v(\sigma))\rangle_{\Lambda_L^2}^{\tau_1} - \langle f(T_v(\sigma))\rangle_{\Lambda_L^2}^{\tau_2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}
$$

For the $d$-dimensional random-field spin $O(n)$ model with $n \geq 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E}\left(\sup_{\tau: \mathbb{Z}^d \to S} \left\| \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle \sigma_v\rangle_{\Lambda_L^d}^{\tau} \right\| \right) \leq C \begin{cases} 
L^{-\frac{1}{3}} & d = 2 \\
L^{-\frac{1}{5}} & d = 3 \\
(log \log L)^{-\frac{1}{2}} & d = 4
\end{cases}
$$
Uniqueness problem

- **Conjecture**: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that $\eta$-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of
  \[ \langle f(\mathcal{T}_v(\sigma)) \rangle_{\mu} \]
  is the same for all Gibbs measures $\mu$ of the $\eta$-disordered system.

- The conjecture is equivalent to the following finite-volume statement:
  \[ \lim_{L \to \infty} \sup_{\tau_1, \tau_2: \mathbb{Z}^2 \to S} \left\| \langle f(\sigma) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f(\sigma) \rangle_{\Lambda_L^2}^{\tau_2} \right\| = 0, \quad \eta\text{-almost surely} \]

- The value of $\mathcal{T}_v(\sigma)$ itself need not be unique in general systems. For instance, a global sign flip applied to $\sigma$ in a spin glass system (with Hamiltonian $H^\eta(\sigma) = -\lambda \sum_{u \sim v} \eta_{u,v} \sigma_u \sigma_v$) takes one Gibbs measure to another.

- Applied to two-dimensional spin glasses at zero temperature, the conjecture implies the conjecture that the spin glass system has a unique ground-state pair.
Partial uniqueness result

- Due to the disorder in the systems considered, it does not make sense to consider translation-invariant Gibbs measures. Instead, the following notion of translation-covariant Gibbs measures has been proposed.
- A measurable map $\rho$ from the disorder variables $\eta$ to the Gibbs measures of the $\eta$-disordered system is called a translation-covariant Gibbs measure if $\rho(T_v(\eta)) = T_v(\rho(\eta))$ for all vertices $v \in \mathbb{Z}^d$ (the translation $T_v$ naturally extends to Gibbs measures).
- Compactness arguments (Aizenman-Wehr, Newman-Stein) show that translation-covariant Gibbs measures always exist for the disordered systems considered above (as barycenters of translation-covariant metastates).
- **Theorem**: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that $\eta$-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of $\langle f(T_v(\sigma)) \rangle_{\rho(\eta)}$ is the same for all translation-covariant Gibbs measures $\rho$.
- **Corollary**: For the two-dimensional spin glass model at zero temperature, if there exists a translation-covariant extremal Gibbs measure then there is a unique translation-covariant Gibbs measure up to a global sign flip.
Proof sketch for compact state space

• Theorem recalled: For the above disordered systems with compact state space in two dimensions, at $0 \leq T < \infty$ and $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E} \left( \sup_{\tau_1, \tau_2 : \mathbb{Z}^2 \to S} \left\| \frac{1}{L^2} \sum_{\nu \in \Lambda_L^2} \langle f(T_\nu(\sigma)) \rangle^\tau_1_{\Lambda_L^2} - \langle f(T_\nu(\sigma)) \rangle^\tau_2_{\Lambda_L^2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}
$$

• To simplify, assume $f(\sigma) = f(\sigma_0) \in \mathbb{R}$ and fix $T > 0$. Write $Z^\eta_{T, \Lambda, \tau}$ for the partition function at temperature $T$, in a finite $\Lambda \subset \mathbb{Z}^2$ and with boundary conditions $\tau$. Thus

$$
Z^\eta_{T, \Lambda, \tau} := \int e^{-\frac{1}{T}H^\eta_\Lambda(\sigma)} \prod_{\nu \in \Lambda} d\kappa(\sigma_\nu) \prod_{\nu \in \Lambda^c} \delta_{\tau_\nu}(\sigma_\nu)
$$

with $H^\eta_\Lambda(\sigma)$ the terms in the Hamiltonian $H^\eta(\sigma) = H(\sigma) - \lambda \sum_\nu \eta_\nu f(T_\nu(\sigma))$ depending on the spins in $\Lambda$. Let $F^\eta_\Lambda(\tau) := \frac{T}{|\Lambda|} \log Z^\eta_{T, \Lambda, \tau}$ be minus the free energy.

• Standard facts: 1) $F^\eta_\Lambda(\tau)$ is a convex function of $\eta$.

• 2) For each $\Lambda$: $\sup_{\tau_1, \tau_2} \left| F^\eta_\Lambda(\tau_1) - F^\eta_\Lambda(\tau_2) \right| \leq \frac{C|\partial \Lambda|}{|\Lambda|}$.

• 3) Write $\eta = (\hat{\eta}_\Lambda, \eta^\perp_\Lambda)$ where $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|} \sum_{\nu \in \Lambda} \eta_\nu$ and $\eta^\perp_\Lambda, \nu := \eta_\nu - \hat{\eta}_\Lambda$. Then

$$
\frac{\partial}{\partial \hat{\eta}_\Lambda} F^{(\hat{\eta}_\Lambda, \eta^\perp_\Lambda)}_\Lambda(\tau) = \frac{\lambda}{|\Lambda|} \sum_{\nu} \langle f(T_\nu(\sigma)) \rangle^\tau_\Lambda, \text{ with the sum over terms involving spins in } \Lambda
$$
Proof sketch II

- **Lemma:** Let $\Lambda$ satisfy $|\partial\Lambda| \leq C\sqrt{|\Lambda|}$. Then for each $\delta > 0$,

$$\mathbb{P}\left( \sup_{\tau_1, \tau_2: \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda|} \sum_v f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_1} \right) \right) - f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_2} \right) \right) \right| < 2\delta \right) \geq \exp \left( - \frac{C \lambda^2}{\delta^4} \right)$$

- **Proof sketch:** Claim: Let $g: \mathbb{R} \to \mathbb{R}$ be a convex 1-Lipschitz function. Set $N_r(g) := \{ h: \mathbb{R} \to \mathbb{R} \text{ convex 1–Lipschitz } \ | \ |h - g|_\infty \leq r \}$. Then for each $r, \delta > 0$.

$$\text{Leb}(\{ x \in \mathbb{R} \mid \exists h \in N_r(f), |h'(x) - g'(x)| \geq \delta \}) \leq \frac{Cr}{\delta^2}$$

- Fix $\tau_0: \mathbb{Z}^2 \to S$ and let $g(x) := F^{(x, \eta^\Lambda)}_{\Lambda}(\tau_0)$. Then for all $\tau$, $F^{(\cdot, \eta^\Lambda)}_{\Lambda}(\tau) \in N_{C|\partial\Lambda|/|\Lambda|}(g)$. Thus, the Claim implies that

$$\text{Leb} \left( \left\{ x \in \mathbb{R} \mid \exists \tau: \mathbb{Z}^2 \to S, \left| \frac{\partial}{\partial \eta^\Lambda} g^{(x, \eta^\Lambda)}_{\Lambda}(\tau) - \frac{\partial}{\partial \eta^\Lambda} g^{(x, \eta^\Lambda)}_{\Lambda}(\tau_0) \right| \geq \delta \right\} \right) \leq \frac{C \lambda |\partial\Lambda|}{|\Lambda| \delta^2} \leq \frac{C \lambda}{\sqrt{|\Lambda|} \delta^2}$$

- Since $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|} \sum_{v \in \Lambda} \eta_v$ is Gaussian with standard deviation $\frac{1}{\sqrt{|\Lambda|}}$ we conclude that

$$\mathbb{P}\left( \sup_{\tau: \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda|} \sum_v f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau} \right) \right) - f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_0} \right) \right) \right| < \delta \right) \geq \exp \left( - \frac{C \lambda^2}{\delta^4} \right)$$

which implies the lemma.
Proof sketch III

• Let \( L \geq 2 \). Call a set \( \Lambda' \subset \Lambda_L \) \( \varepsilon \)-fluctuative if

\[
\sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda'|} \sum_v f \left( T_v \left( \sigma_{\Lambda', \tau_1}^\eta \right) \right) - f \left( T_v \left( \sigma_{\Lambda', \tau_2}^\eta \right) \right) \right| < \varepsilon
\]

• Perform a fractal percolation: Set \( \delta := \frac{c \sqrt{\lambda}}{(\log \log L)^4} \) and \( k = C \lambda / \delta \).

Partition \( \Lambda_L \) into \( k \) squares. Then partition each of these into \( k \) squares and so on until reaching squares of constant size. A square in this recursive partition is taken if it is \( 4\delta \)-fluctuative and the squares containing it are not \( 4\delta \)-fluctuative.

• Define \( B := \{ v \in \Lambda_L \mid v \text{ is not in a taken square} \} \). Then

\[
\sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda_L|} \sum_v f \left( T_v \left( \sigma_{\Lambda_L, \tau_1}^\eta \right) \right) - f \left( T_v \left( \sigma_{\Lambda_L, \tau_2}^\eta \right) \right) \right| \leq 4\delta + \frac{C|B|}{|\Lambda_L|}
\]

• It remains to show that \( \mathbb{P}(v \in B) \leq \delta \). Write \( \Lambda_0(v) \supset \Lambda_1(v) \supset \Lambda_2(v) \supset \cdots \) for the partition squares containing \( v \). Since \( |\Lambda_{\ell+1}(v)| \leq c \delta |\Lambda_\ell(v)| / \lambda \), one concludes that

\[
\{ v \in B \} \subset \bigcap_{\ell} \{ \Lambda_\ell(v) \setminus \Lambda_{\ell+1}(v) \text{ is not } 2\delta-\text{fluctuative} \}
\]

• The events in the intersection are independent since the annuli are disjoint. The previous lemma bounds their probabilities, concluding the proof.
Non-compact case: Random-field random surfaces

- We now discuss the effect of disorder on systems with non-compact state space. Our focus is on random surface models.
- Let $(\eta_v)_{v \in \mathbb{Z}^d}$ be independent standard Gaussian random variables.
- A real-valued random-field random surface is the model on $\phi: \mathbb{Z}^d \to \mathbb{R}$ with Hamiltonian

$$H^\eta(\phi) = \sum_{u \sim v} V(\phi_u - \phi_v) - \lambda \sum_v \eta_v \phi_v$$

where $V: \mathbb{R} \to \mathbb{R}$ is a measurable even function termed the potential. The case $V(x) = x^2$ is the real-valued random-field Gaussian free field.
- We also study the integer-valued random-field Gaussian free field which has the same Hamiltonian as above with $V(x) = x^2$ but restricts to $\phi: \mathbb{Z}^d \to \mathbb{Z}$.
- Our goal the localization/delocalization properties of these disordered surfaces.
- Without disorder: the gradient of these surfaces localizes in all dimensions $d \geq 1$. On $\Lambda_L^d$, real-valued surfaces delocalize with variance $L$ when $d = 1$ and with variance $\log L$ when $d = 2$ while staying localized for $d \geq 3$. The integer-valued GFF behaves similarly except for a roughening transition when $d = 2$, from localized to logarithmic delocalization as the temperature increases.
Theorem (Dario-Harel-P 2020+): Consider the real-valued random-field random surfaces above at all temperatures $0 \leq T < \infty$ and all disorder strengths $\lambda > 0$ on $\Lambda^d_L$ with zero boundary conditions. Assume $0 < c_- \leq V'' \leq c_+ < \infty$. Then

- Discrete Gradient: $\mathbb{E} \left( \left( \frac{1}{L^d} \sum_{\{u,v\} \in \mathcal{E}(\Lambda^d_L)} (\phi_u - \phi_v)^2 \right) \right) \approx \begin{cases} L & d = 1 \\ \log L & d = 2 \\ 1 & d \geq 3 \end{cases}$

- Height fluctuations: $\mathbb{E} \left( \langle \phi_0 \rangle^2 \right) \approx \begin{cases} L^{4-d} & d = 1, 2, 3 \\ \log L & d = 4 \\ 1 & d \geq 5 \end{cases}$

Theorem (Dario-Harel-P 2020+): The integer-valued random-field Gaussian free field, at all temperatures $0 \leq T < \infty$ and disorder strengths $\lambda > 0$, satisfies the gradient estimate above, and, when $d = 1, 2$, satisfies

$$\mathbb{E} \left( \left( \frac{1}{L^d} \sum_{v \in \Lambda^d_L} \phi_v^2 \right) \right) \approx L^{4-d}$$

Additionally, this expectation is bounded in $L$ in dimensions $d \geq 3$ at low temperatures and small disorder strength $\lambda > 0$. 
Random-field random surfaces: previous results

- Bovier-Külske studied a random field Solid-On-Solid model in which the disorder enters differently from the way it is introduced here. They proved a certain form of delocalization in two dimensions (Bovier-Külske 1996) and localization in three and higher dimensions (Bovier-Külske 1994).

- Külske and Orlandi 2006 prove that for all deterministic fields $\eta$, a random surface with field $\eta$ will delocalize with at least logarithmic variance in two dimensions, when the potential $V$ satisfies $\sup V(x) < \infty$.

- Van Enter and Külske 2008 proved a form of delocalization for the gradients of the random-field random surface for a wide class of potentials in two dimensions. The result is non-quantitative. They further proved a lower bound on the rate of correlation decay for gradient Gibbs measures, when they exist, in three dimensions.

- Cotar and Külske proved the existence of translation-covariant gradient Gibbs measures for random-field random surfaces in dimensions $d \geq 3$ (Cotar and Külske 2012) and their uniqueness for each given expected tilt (Cotar and Külske 2015), for a large class of potentials.

- Later results: Dario 2023 (thermodynamic limit), Sakagawa 2023 (maximum).
Open questions

• For disordered systems with compact state space, improve the bounds on

\[
\mathbb{E} \left( \sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left\| \frac{1}{L^d} \sum_{\nu \in \Lambda_L^d} f(T_\nu(\sigma)) \right\|_{\Lambda_L^d}^{\tau_1} - f(T_\nu(\sigma)) \right\|_{\Lambda_L^d}^{\tau_2} \right)
\]

If the sum is performed over a concentric box of half the size, does it decay exponentially fast with \(L\) in two dimensions at all \(T\) and \(\lambda > 0\)?

• **Uniqueness conjecture:** For two-dimensional disordered systems, for each \(\nu \in \mathbb{Z}^2\), \(\eta\)-almost surely, the value of \(\langle f(T_\nu(\sigma)) \rangle_\mu\) is the same for all Gibbs measures \(\mu\).

• Is there a Berezinskii-Kosterlitz-Thouless type transition as the disorder strength lowers (i.e., transition from exponential to power-law decay) for the random-field spin \(O(n)\) models with \(n = 2\) in dimensions \(d = 3\) or \(d = 4\)? What about \(n \geq 3\)?

• What is the localization/delocalization behavior of the integer-valued random-field Gaussian free field in dimensions \(d \geq 3\) at high disorder strength \(\lambda\)?

**Conjecture:** Delocalization in dimension \(d = 3\) and localization when \(d \geq 5\). Thus we conjecture a roughening transition in the disorder strength for \(d = 3\).