Quantitative estimates for the effect of disorder on low-dimensional lattice systems

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Lattice systems with compact state space

• We discuss statistical physics systems on $\mathbb{Z}^d$, aiming to develop a quantitative understanding of the effect of adding disorder to them.

• We start with the case of a compact state space.

• Setup: (1) Compact metric space $S$ equipped with a Borel measure $\kappa$.
  (2) Translation-invariant finite range and finite energy Hamiltonian $H$.

• As usual, for a finite domain $\Lambda \subset \mathbb{Z}^d$, at temperature $T$ and with boundary conditions $\tau: \mathbb{Z}^d \to S$, configurations $\sigma: \mathbb{Z}^d \to S$ coinciding with $\tau$ outside $\Lambda$ are sampled from the probability measure with density

$$\frac{1}{Z_{T,\Lambda,\tau}} \exp \left( -\frac{1}{T} H_\Lambda (\sigma) \right)$$

with respect to the measure $\prod_v d\kappa(\sigma_v)$, where $Z_{T,\Lambda,\tau}$ is the partition function and $H_\Lambda$ contains the terms in the Hamiltonian depending on the spins in $\Lambda$.

Periodic boundary conditions and the zero-temperature limit are also allowed.

• Examples: Ising model: $S = \{-1,1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$

• Potts model: $S = \{1,2, ..., q\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} 1_{\sigma_u = \sigma_v}$

• Spin O($n$) model with $n \geq 2$: $S = S^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2$
Disordered lattice systems

• Noised observables: Let \( f : S^{\mathbb{Z}^d} \to \mathbb{R}^m \), for some \( m \geq 1 \), be a bounded measurable function depending on the spins in a finite neighborhood of the origin.

Disorder: Let \( (\eta_v)_{v \in \mathbb{Z}^d} \) be independent standard \( m \)-dimensional Gaussian vectors.

Disordered Hamiltonian: \( H^\eta(\sigma) = H(\sigma) - \lambda \sum_v \eta_v \cdot f(\mathcal{T}_v(\sigma)) \) where \( \mathcal{T}_v(\sigma) \) is the configuration \( \sigma \) translated by \( v \).

• Examples: Random-field Ising model: \( m = 1 \) and \( f(\sigma) = \sigma_0 \). Thus

\[
H^\eta(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v
\]

• Edwards-Anderson spin glasses: \( S = \{-1, 1\}, \mu = \text{counting}, f(\sigma) = (\sigma_{e_j} \sigma_0)_{j=1}^d \).

\[
H^\eta(\sigma) = -\lambda \sum_{u \sim v} \eta_{u,v} \sigma_u \sigma_v
\]

• Random-field \( q \)-state Potts model: \( m = q \) and \( f(\sigma) = (1_{\sigma_0=1}, \ldots, 1_{\sigma_0=q}) \). Thus

\[
H^\eta(\sigma) = -\sum_{u \sim v} 1_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_{k=1}^q \eta_v, k 1_{\sigma_v=k}
\]

• Random-field spin \( O(n) \) model, \( n \geq 2 \): \( m = n \) and \( f(\sigma) = \sigma_0 \) (with \( S^{n-1} \subset \mathbb{R}^n \)),

\[
H^\eta(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2 - \lambda \sum_v \eta_v \cdot \sigma_v
\]
Imry-Ma phenomenon

• Imry-Ma (1975) considered the effects of disorder for the random-field Ising and spin $O(n)$ models, and predicted that in low dimensions, an arbitrarily small disorder strength $\lambda$ causes the models to lose their ordered phase, as follows: The random-field Ising model is disordered at all temperatures for $d \leq 2$. The random-field spin $O(n)$ model is disordered at all temperatures for $d \leq 4$.

• Aizenman-Wehr (1989) proved the predictions as part of a general statement.

• Notation: Write $\Lambda^d_L := \{-L, \ldots, L\}^d$. For each disorder $\eta$, write $\langle \cdot \rangle_\mu$ for the thermal expectation according to a Gibbs measure $\mu$ of the $\eta$-disordered system. Write $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation operator over $\eta$.

• Theorem (Aizenman-Wehr, special case): For a disordered lattice system with compact state space (as discussed above) in dimensions $d = 1, 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, the limit

$$\lim_{L \to \infty} \frac{1}{L^d} \sum_{v \in \Lambda^d_L} \langle f(T_v(\sigma)) \rangle_\mu$$

exists and has the same value for all Gibbs measures $\mu$ and almost all $\eta$. The same holds in dimensions $1 \leq d \leq 4$ for the spin $O(n)$ models with $n \geq 2$.

• Our goal: Develop a quantitative understanding of this phenomenon.
Random-field Ising model

• Random-field Ising model Hamiltonian: $H^\eta(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$

• The disordered model satisfies the usual monotonicity (FKG) properties. In particular, the model has maximal and minimal Gibbs measures $\mu^{\eta,+}$ and $\mu^{\eta,-}$, arising in the thermodynamic limit from constant boundary conditions. The Aizenman-Wehr theorem implies that $\mu^{\eta,+} = \mu^{\eta,-}$ in two dimensions $\eta$-almost surely, so that the model has a unique Gibbs measure.

• A natural quantitative parameter is $m_L := \mathbb{E} \left( \langle \sigma_0 \rangle^+_{\Lambda_L^2} \right)$ where $\langle \cdot \rangle^+_{\Lambda_L^2}$ denotes the thermal expectation in $\{-L, ..., L\}^2$ with $+1$ boundary conditions.

• A bound of the form $m_L \leq \exp(-c(\lambda,T)L)$ is relatively simple for large disorder strength $\lambda$ or high temperatures $T$, so interested in small $\lambda$ and low temperature.

• Results: $m_L \leq \frac{C(\lambda)}{\sqrt{\log \log L}}$ (Chatterjee 2017), $m_L \leq \frac{C(\lambda)}{L^c(\lambda)}$ (Aizenman-P. 2018) and finally $m_L \leq C(\lambda) \exp(-c(\lambda)L)$ proved at zero temperature by Ding-Xia 2019 and then at positive temperature by Ding-Xia 2019 and Aizenman-Harel-P. 2019.

• Still open to determine correlation length $c(\lambda)$. Proof seems to yield $c(\lambda) \leq e^{e^{1/\lambda^2}}$ while physics predictions are that $c(\lambda) \approx e^{1/\lambda}$ or $c(\lambda) \approx e^{1/\lambda^2}$. 
Quantitative results

• The other models discussed (Potts, spin-glasses, spin $O(n)$) do not share the monotonicity properties of the random-field Ising model and the proof techniques break down for them. Indeed, even the choice of which quantity to bound is non-obvious since it is unclear which boundary conditions $\tau$ maximize or minimize the average $\langle f(T_v(\sigma)) \rangle_{\Lambda_L^2}^\tau$ and, indeed, it may be that these boundary conditions depend on the disorder $\eta$ and on $L$ and $v$. We obtain the following results.

• Theorem (Dario-Harel-P 2020+): For each two-dimensional disordered lattice system of the type described above, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E} \left( \sup_{\tau_1, \tau_2: \mathbb{Z}^2 \to S} \left\| \frac{1}{L^2} \sum_{v \in \Lambda_L^2} \langle f(T_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f(T_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}
$$

For the $d$-dimensional random-field spin $O(n)$ model with $n \geq 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E} \left( \sup_{\tau: \mathbb{Z}^d \to S} \left\| \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle f(T_v(\sigma)) \rangle_{\Lambda_L^d}^{\tau} \right\| \right) \leq C \begin{cases} 
L^{-\frac{1}{3}} & d = 2 \\
L^{-\frac{1}{5}} & d = 3 \\
(\log \log L)^{-\frac{1}{2}} & d = 4 
\end{cases}
$$
Uniqueness problem

• **Conjecture**: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that $\eta$-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of

\[
\langle f(T_v(\sigma)) \rangle_\mu
\]

is the same for all Gibbs measures $\mu$ of the $\eta$-disordered system.

• The conjecture is equivalent to the following **finite-volume statement**:

\[
\lim_{L \to \infty} \sup_{\tau_1, \tau_2 : \mathbb{Z}^2 \to S} \left\| \langle f(\sigma) \rangle_{\Lambda_L^{\tau_1}}^{\tau_1} - \langle f(\sigma) \rangle_{\Lambda_L^{\tau_2}}^{\tau_2} \right\| = 0, \quad \eta-\text{almost surely}
\]

• The value of $T_v(\sigma)$ itself need not be unique in general systems. For instance, a global sign flip applied to $\sigma$ in a spin glass system (with Hamiltonian $H^\eta(\sigma) = -\lambda \sum_{u, v} \eta_{u, v} \sigma_u \sigma_v$) takes one Gibbs measure to another.

• Applied to two-dimensional spin glasses at zero temperature, the conjecture implies the conjecture that the spin glass system has a unique ground-state pair.
Partial uniqueness result

• Due to the disorder in the systems considered, it does not make sense to consider translation-invariant Gibbs measure. Instead, the following notion of translation-covariant Gibbs states has been proposed.

• A measurable map $\rho$ from the disorder variables $\eta$ to the Gibbs measures of the $\eta$-disordered system is called a translation-covariant Gibbs measure if

$$\rho(T_v(\eta)) = T_v(\rho(\eta))$$

for all vertices $v \in \mathbb{Z}^d$ (the translation $T_v$ naturally extends to Gibbs measures).

• Compactness arguments (Aizenman-Wehr, Newman-Stein) show that translation-covariant Gibbs measures always exist for the disordered systems considered above (as barycenters of translation-covariant metastates).

• **Theorem**: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that $\eta$-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of

$$\langle f(T_v(\sigma)) \rangle_{\rho(\eta)}$$

is the same for all translation-covariant Gibbs measures $\rho$.

• **Corollary**: For the two-dimensional spin glass model at zero temperature, if there exists a translation-covariant extremal Gibbs measure then there is a unique translation-covariant Gibbs measure up to a global sign flip.
Proof sketch for compact state space

- **Theorem recalled:** For the above disordered systems with compact state space in two dimensions, at $0 \leq T < \infty$ and $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$
\mathbb{E}\left( \sup_{\tau_1, \tau_2 : \mathbb{Z}^2 \to S} \left\| \frac{1}{L^2} \sum_{v \in \Lambda_L^2} \langle f (\mathcal{T}_v (\sigma)) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f (\mathcal{T}_v (\sigma)) \rangle_{\Lambda_L^2}^{\tau_2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}
$$

- To simplify, assume $f$ is scalar and fix $T > 0$. Write $Z_{T, \Lambda, \tau}^\eta$ for the partition function at temperature $T$, in a finite $\Lambda \subset \mathbb{Z}^2$ and with boundary conditions $\tau$. Thus

$$
Z_{T, \Lambda, \tau}^\eta := \int e^{-\frac{1}{T}H_{\Lambda}^\eta (\sigma)} \prod_{v \in \Lambda} d\kappa (\sigma_v) \prod_{v \in \Lambda^c} \delta_{\tau_v} (\sigma_v)
$$

with $H_{\Lambda}^\eta (\sigma)$ the terms in the Hamiltonian $H_{\Lambda}^\eta (\sigma) = H (\sigma) - \lambda \sum_v \eta_v f (\mathcal{T}_v (\sigma))$ depending on the spins in $\Lambda$. Let $F_{\Lambda}^\eta (\tau) := \frac{T}{|\Lambda|} \log Z_{T, \Lambda, \tau}^\eta$ be minus the free energy.

- **Standard facts:**
  1) $F_{\Lambda}^\eta (\tau)$ is a convex function of $\eta$.
  2) For each $\Lambda$, with high $\eta$ probability: $\sup_{\tau_1, \tau_2} |F_{\Lambda}^\eta (\tau_1) - F_{\Lambda}^\eta (\tau_2)| \leq \frac{C|\partial \Lambda|}{|\Lambda|}$.
  3) Write $\eta = (\hat{\eta}_\Lambda, \eta^\perp_\Lambda)$ where $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|} \sum_{v \in \Lambda} \eta_v$ and $\eta^\perp_\Lambda := \eta_v - \hat{\eta}_\Lambda$. Then

$$
\frac{\partial}{\partial \hat{\eta}_\Lambda} F_{\Lambda}^{(\hat{\eta}_\Lambda, \eta^\perp_\Lambda)} (\tau) = \frac{\lambda}{|\Lambda|} \sum_v \langle f (\mathcal{T}_v (\sigma)) \rangle_{\Lambda}^\tau ,
$$

with the sum over terms involving spins in $\Lambda$. 

Proof sketch II

- **Lemma:** Let $\Lambda$ satisfy $|\partial \Lambda| \leq C \sqrt{|\Lambda|}$. Then for each $\delta > 0$,

$$
\mathbb{P} \left( \sup_{\tau_1, \tau_2: \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda|} \sum_{v} f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_1} \right) \right) - f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_2} \right) \right) \right| < 2\delta \right) \geq \exp \left( -\frac{C}{\delta^4} \right)
$$

- **Proof sketch:** **Claim:** Let $g: \mathbb{R} \to \mathbb{R}$ be a convex 1-Lipschitz function. Set $N_r(g) := \{ h: \mathbb{R} \to \mathbb{R} \text{ convex 1–Lipschitz} \mid \| h - g \|_\infty \leq r \}$. Then for each $r, \delta > 0$.

$$
\text{Leb}(\{ x \in \mathbb{R} \mid \exists h \in N_r(f), |h'(x) - g'(x)| \geq \delta \}) \leq \frac{Cr}{\delta^2}
$$

- Fix $\tau_0: \mathbb{Z}^d \to S$ and let $g(x) := F_{\Lambda}^{(x, \eta_{\Lambda_L}^L)}(\tau_0)$. Then for all $\tau$, $F_{\Lambda}^{(x, \eta_{\Lambda_L}^L)}(\tau) \in N_{C|\partial \Lambda|/|\Lambda|}(g)$ (with high $\eta$ probability). On this event, the Claim implies that

$$
\text{Leb} \left( \left\{ x \in \mathbb{R} \mid \exists \tau: \mathbb{Z}^d \to S, \left| \frac{\partial}{\partial \hat{\eta}_{\Lambda}} \varepsilon^{(x, \eta_{\Lambda_L}^L)}(\tau) - \frac{\partial}{\partial \hat{\eta}_{\Lambda}} \varepsilon^{(x, \eta_{\Lambda_L}^L)}(\tau_0) \right| \geq \delta \right\} \right) \leq \frac{C|\partial \Lambda|}{|\Lambda|\delta^2} \leq \frac{C}{\sqrt{|\Lambda|}\delta^2}
$$

- Since $\hat{\eta}_{\Lambda} := \frac{1}{|\Lambda|} \sum_{v \in \Lambda} \eta_v$ is Gaussian with standard deviation $\frac{1}{\sqrt{|\Lambda|}}$ we conclude that

$$
\mathbb{P} \left( \sup_{\tau: \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda|} \sum_{v} f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau} \right) \right) - f \left( \mathcal{T}_v \left( \sigma^\eta_{\Lambda, \tau_0} \right) \right) \right| < \delta \right) \geq \exp \left( -\frac{C}{\delta^4} \right)
$$

which implies the lemma.
Proof sketch III

- Let $L \geq 2$. Call a set $\Lambda' \subset \Lambda_L$ $\epsilon$-fluctuative if
  \[
  \sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda'|} \sum_v f \left( T_v \left( \sigma_{\Lambda', \tau_1}^\eta \right) \right) - f \left( T_v \left( \sigma_{\Lambda', \tau_2}^\eta \right) \right) \right| < \epsilon
  \]
- Perform a Mandelbrot percolation: Set $\delta := \frac{C}{(\log \log L)^4}$ and $k = C/\delta$.
  Partition $\Lambda_L$ into $k$ squares. Then partition each of these into $k$ squares and so on until reaching squares of constant size. A square in this recursive partition is taken if it is $4\delta$-fluctuative and the squares containing it are not $4\delta$-fluctuative.
- Define $B := \{ v \in \Lambda_L \mid v \text{ is not in a taken square} \}$. Then
  \[
  \sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left| \frac{\lambda}{|\Lambda_L|} \sum_v f \left( T_v \left( \sigma_{\Lambda_L, \tau_1}^\eta \right) \right) - f \left( T_v \left( \sigma_{\Lambda_L, \tau_2}^\eta \right) \right) \right| \leq 4\delta + \frac{C|B|}{|\Lambda_L|}
  \]
- It remains to show that $\mathbb{P}(v \in B) \leq \delta$. Write $\Lambda_0(v) \supset \Lambda_1(v) \supset \Lambda_2(v) \supset \cdots$ for the partition squares containing $v$. Since $|\Lambda_{\ell+1}(v)| \leq c\delta|\Lambda_{\ell}(v)|$, one concludes that
  \[
  \{ v \in B \} \subset \bigcap_{\ell} \{ \Lambda_{\ell}(v) \setminus \Lambda_{\ell+1}(v) \text{ is not } 2\delta \text{-fluctuative} \}
  \]
- The events in the intersection are independent since the annuli are disjoint. The previous lemma bounds their probabilities, concluding the proof.
Non-compact case: Random-field random surfaces

- We now discuss the effect of disorder on systems with non-compact state space. Our focus is on random surface models.
- Let $(\eta_v)_{v \in \mathbb{Z}^d}$ be independent standard Gaussian random variables.
- A real-valued random-field random surface is the model on $\phi: \mathbb{Z}^d \to \mathbb{R}$ with Hamiltonian

$$H_\eta(\sigma) = \sum_{u \sim v} V(\phi_u - \phi_v) - \lambda \sum_v \eta_v \phi_v$$

where $V: \mathbb{R} \to \mathbb{R}$ is a measurable even function termed the potential. The case $V(x) = x^2$ is the real-valued random-field Gaussian free field.
- We also study the integer-valued random-field Gaussian free field which has the same Hamiltonian as above with $V(x) = x^2$ but restricts to $\phi: \mathbb{Z}^d \to \mathbb{Z}$.
- Our goal the localization/delocalization properties of these disordered surfaces.
- Without disorder: the gradient of these surfaces localizes in all dimensions $d \geq 1$. On $\Lambda_L^d$, real-valued surfaces delocalize with variance $L$ when $d = 1$ and with variance $\log L$ when $d = 2$ while staying localized for $d \geq 3$. The integer-valued GFF behaves similarly except for a roughening transition when $d = 2$, from localized to logarithmic delocalization as the temperature increases.
Random-field random surfaces: results

• **Theorem (Dario-Harel-P 2020+):** Consider the real-valued random-field random surfaces above at all temperatures $0 \leq T < \infty$ and all disorder strengths $\lambda > 0$ on $\Lambda^d_L$ with zero boundary conditions. Assume the potential $V$ is $C^2$ and satisfies $0 < c_- \leq V'' \leq c_+ < \infty$

  – The gradient of the random surface satisfies
    \[
    \mathbb{E} \left( \left\| \frac{1}{L^d} \sum_{e \in E(\Lambda^d_L)} (\nabla \phi(e))^2 \right\| \right) \approx \begin{cases} 
    \frac{L}{\log L} & d = 1 \\
    1 & d \geq 3 
  \end{cases}
  \]

  – The random-field random surface satisfies
    \[
    \mathbb{E}(\langle \phi(0) \rangle) \approx \begin{cases} 
    \frac{L^{4-d}}{\log L} & d = 1,2,3 \\
    1 & d = 4,5 
  \end{cases}
  \]

• **Theorem (Dario-Harel-P 2020+):** The integer-valued random-field Gaussian free field, at all temperatures $0 \leq T < \infty$ and disorder strengths $\lambda > 0$, satisfies the gradient estimate above, and, when $d = 1,2$, satisfies
    \[
    \mathbb{E} \left( \left\| \frac{1}{L^d} \sum_{v \in \Lambda^d_L} \phi(v)^2 \right\| \right) \approx L^{4-d}
    \]
Random-field random surfaces: previous results

- **Bovier-Krülske** studied a random field Solid-On-Solid model in which the disorder enters differently from the way it is introduced here. They proved a certain form of delocalization in two dimensions (**Bovier-Krülske 1996**) and localization in three and higher dimensions (**Bovier-Krülske 1994**).

- **Külske and Orlandi 2006** prove that for all deterministic fields $\eta$, a random surface with field $\eta$ will delocalize with at least logarithmic variance in two dimensions, when the potential $V$ satisfies $\sup V(x) < \infty$.

- **Van Enter and Külske 2008** proved a form of delocalization for the gradients of the random-field random surface for a wide class of potentials in two dimensions. The result is non-quantitative. They further proved a lower bound on the rate of correlation decay for gradient Gibbs measures, when they exist, in three dimensions.

- **Cotar and Külske** proved the existence of translation-covariant gradient Gibbs measures for random-field random surfaces in dimensions $d \geq 3$ (**Cotar and Külske 2012**) and their uniqueness for each given expected tilt (**Cotar and Külske 2015**), for a large class of potentials.
Open questions

• For disordered systems with compact state space, improve the bounds on

$$\mathbb{E} \left( \sup_{\tau_1, \tau_2 : \mathbb{Z}^d \to S} \left\| \frac{1}{L^d} \sum_{v \in \Lambda_L} \langle f(T_v(\sigma)) \rangle_{\tau_1} - \langle f(T_v(\sigma)) \rangle_{\tau_2} \right\| \right)$$

Does it decay \textbf{exponentially fast} with $L$ in two dimensions at all $T$ and $\lambda > 0$?

• \textbf{Uniqueness conjecture:} For two-dimensional disordered systems, for each $v \in \mathbb{Z}^2$, $\eta$-almost surely, the value of $\langle f(T_v(\sigma)) \rangle_{\mu}$ is the same for all Gibbs measures $\mu$.

• Is there a \textbf{Berezinskii-Kosterlitz-Thouless} type transition as the disorder strength lowers (i.e., transition from \textbf{exponential to power-law decay}) for the random-field spin $O(n)$ models with $n = 2$ in dimensions $d = 3$ or $d = 4$? What about $n \geq 3$?

• \textbf{What is the localization/delocalization} behavior of the \textbf{integer-valued} random-field Gaussian free field in dimensions $d \geq 3$? We conjecture that it always delocalizes when $d = 3$ and localizes when $d \geq 5$. Is there a \textbf{roughening transition} as the temperature or disorder strength changes in dimension $d = 4$? We speculate that it always delocalizes when $d = 4$. The problem is related to the behavior of the \textbf{random-phase sine-Gordon model}. 