Gravitational allocation to Poisson points

Ron Peled

joint work with

Sourav Chatterjee
Yuval Peres
Dan Romik
Let $\Xi$ be a discrete subset of $\mathbb{R}^d$. 

For $z \in \Xi$, we call $\psi^{-1}(z)$ the cell allocated to $z$. 

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Allocation rules

- Let $\Xi$ be a discrete subset of $\mathbb{R}^d$.
- An allocation (of Lebesgue measure to $\Xi$) is a measurable function $\psi : \mathbb{R}^d \to \Xi \cup \{\infty\}$ that satisfies
  
  \[
  \text{Vol}(\psi^{-1}(\infty)) = 0, \\
  \text{Vol}(\psi^{-1}(z)) = 1, \quad z \in \Xi,
  \]

  where $\text{Vol}(\cdot)$ is Lebesgue measure in $\mathbb{R}^d$. 

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Gravitational allocation to Poisson points
Figure: (a) The two-dimensional stable marriage allocation for a Poisson process (picture due to Alexander E. Holroyd). (b) The gradient flow allocation (picture due to Manjunath Krishnapur).
Let $Z$ be a translation-invariant simple point process in $\mathbb{R}^d$ with unit intensity.
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An **equivariant allocation rule** (of Lebesgue measure to $Z$) is a measurable mapping $Z \rightarrow \psi_Z$ such that:

1. a.s. $\psi_Z$ is an allocation of Lebesgue measure to $Z$,
2. the mapping $Z \rightarrow \psi_Z$ is translation-equivariant, i.e. $\psi_Z(x + y) \equiv \psi_Z(y) + x$.

The rule is called **non-randomized** if it is a function only of $Z$, it is called **randomized** if it uses extra randomness.

If a.s. all the cells are bounded, one can consider the allocation diameter $X = \text{diam}(\psi_Z^{-1}(0))$.

One object of interest: The rate of decay of the tail $P(X > R)$. 

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This problem has been considered by many authors. Interested in minimizing various functionals of the matching such as average edge length or maximal edge length.

In dimension 2, it was shown (Ajtai, Komlós and Tusnády 84) that the minimal average edge length grows at rate $\Theta(\sqrt{\log n})$. The minimal maximal edge length (Leighton and Shor 89) grows at rate $\Theta(\log^{3/4} n)$.

In higher dimensions, minimal average edge length remains bounded. Moreover, Talagrand (94) shows that the average of the function $\exp(c_d x)$ of the edge lengths may remain bounded. More general results exist as well.

Questions about allocations are, in some sense, infinite volume analogues of the matching problem. Results on finite volume matchings can yield existence of good randomized equivariant allocations, but new techniques are required to construct non-randomized equivariant allocations.
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Holroyd & Peres: If $d = 1, 2$ and $Z$ is a standard Poisson point process of unit intensity in $\mathbb{R}^d$, then the allocation diameter of any equivariant rule $X$ satisfies $\mathbb{E}X^{d/2} = \infty$. 

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In the stable marriage allocation, a.s.

1. all the cells are bounded and contain their owners,
2. but not all are connected,
3. and when $Z$ is a Poisson point process the allocation diameter $X$ satisfies $\mathbb{E}X^d = \infty$. 
**Figure:** The 2-diml. stable marriage allocation for a Poisson process

**Construction:** Each point of the process grows a ball at unit rate and captures all the sites it reaches first, until it obtains volume 1.
Gravitational force field due to Poisson points

- Let $\mathcal{Z}$ be a standard Poisson process in $\mathbb{R}^d$ ("the stars"), $d \geq 3$. 

- Based on previous work of Nazarov, Tsirelson, Sodin and Volberg who investigated the "gradient flow allocation" to zeros of the Gaussian entire function, we define the "gravitational allocation" to the Poisson points.

- Consider the random stationary vector field $F: \mathbb{R}^d \to \mathbb{R}^d$ ("the force") defined by

  \[ F(x) := \sum_{z \in \mathcal{Z}, |z-x| \leq |z-x|} \frac{z-x}{|z-x|^d}, \]  

  where the summands are arranged in order of increasing distance from $x$.

- First investigated in work of S. Chandrasekhar. Later work by Heath & Shepp.

- For $d \geq 3$ the force converges a.s. to a continuously differentiable vector field, off the stars.
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$$\dot{\Gamma}(t) = F(\Gamma(t)).$$

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Denote by $\Gamma_x$ the integral curve with initial condition $\Gamma_x(0) = x$. 

To each center $z \in Z$, define its **basin of attraction** $B(z) = \{x \in \mathbb{R}^d : z \in \text{ends at } x\} \cup \{z\}$.

Define the **gravitational allocation rule** $\psi_Z(x) = \{z \in B(z) \text{ for } z \in Z\} \cup \{\infty \in \bigcup_{z \in Z} B(z)\}$. 

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$$B(z) = \{x \in \mathbb{R}^d \setminus Z \mid \Gamma_x(t) \text{ ends at } z\} \cup \{z\}.$$
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Define the **gravitational allocation rule**

$$\psi_{\mathcal{Z}}(x) = \begin{cases} z & x \in B(z) \text{ for } z \in \mathcal{Z}, \\ \infty & x \notin \bigcup_{z \in \mathcal{Z}} B(z). \end{cases}$$
Figure: Simulation of a cell in 3-dimensional gravitational allocation
Changing the order of summation

- Recall the force at $x$ is defined by

$$F(x) := \sum_{z \in \mathbb{Z}, |z-x| \uparrow} \frac{z - x}{|z - x|^d}$$

This series does not converge absolutely!

We will need to differentiate the force field, it will be easier to do so if we summed the stars according to their distance from a fixed point, say the origin.

We find that the difference between the two orders of summation is a.s. a constant times $x$, i.e.

$$F(x) = \sum_{z \in \mathbb{Z}, |z-x| \uparrow} \frac{z - x}{|z - x|^d} + \kappa d x$$

It is not hard to see that equality of expectations hold, then equality of the RV’s is shown by controlling the variance of the difference.
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Why equal area in each basin?

▶ Take a basin of attraction $B(z)$, and a point $x \in \partial B(z)$.

▶ If $n$ is the outward-pointing normal vector at $x$, then by the definition of the basin of attraction, $F(x) \cdot n = 0$.

▶ Thus, the oriented surface integral $\int_{\partial B(z)} F(x) \cdot n \, dS = 0$.

▶ Now $\text{div}(F) = \frac{d\kappa}{d\kappa} - \int_{\sum i=1}^{\infty} \delta z_i$, where $(z_i)$ is the set of zeros.

▶ Thus, by the divergence theorem, $\int_{\partial B(z)} F(x) \cdot n \, dS = \int_{B(z)} \text{div}(F) \, dx = \frac{d\kappa}{d\kappa} - \frac{d\kappa}{d\kappa}$.

▶ Combining, we get $\text{Vol}(B(z)) = 1$. 

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  \[ \int_{\partial B(z)} F(x) \cdot n \, dS = \int_{B(z)} \text{div}(F) \, dx = d\kappa_d \text{Vol}(B(z)) - d\kappa_d. \]
- Combining, we get $\text{Vol}(B(z)) = 1$. 
Theorem
The mapping $\mathcal{Z} \rightarrow \psi_\mathcal{Z}$ is an allocation rule of Lebesgue measure to the Poisson point process $\mathcal{Z}$. Almost surely all the cells $\psi^{-1}(z)$ are bounded. The allocation diameter $X = \text{diam}(\psi^{-1}(\psi(0)))$ satisfies the following tail bounds: In dimensions 4 and higher, we have

$$\mathbb{P}(X > R) \leq C_1 \exp \left[ -c_2 R (\log R) \frac{d-2}{d} \right]$$

for some constants $C_1, c_2 > 0$ (depending on the dimension $d$) and all positive $R$. In dimension 3, for any $\alpha > 0$ there exist constants $C_1, c_2 > 0$ (depending on $\alpha$) such that for all $R > 0$ we have

$$\mathbb{P}(X > R) \leq C_1 \exp \left[ -c_2 \frac{R}{(\log R)^{\frac{4}{3} + \alpha}} \right].$$
In a sequel work which is in writing, we give a lower bound for the tail of the diameter, identifying the characteristic exponent in the probability,

**Theorem**

*For all* $d \geq 3$, 

$$
\mathbb{P}(X > R) = \exp(-R^{1+o(1)}) \text{ as } R \to \infty
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Characteristic exponents

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$$

Also, for $Y = |\psi(0)|$, the typical allocation distance, we show

**Theorem**

*For all* $d \geq 3$,

$$
\mathbb{P}(Y > R) = \exp(-R^{\beta_d+o(1)}) \text{ as } R \to \infty
$$

where $\beta_3 = 1$ and $\beta_d = 1 + \frac{1}{d-1}$ for $d \geq 4$. 

S. Chatterjee, R. Peled, Y. Peres, D. Romik

Gravitational allocation to Poisson points
All the previous results are corollaries of the following theorem. Let $Z_R$ be the volume of the cell of the origin after a ball of radius $R$ around $\psi(0)$ was removed from it. Then

**Theorem**

*For all $d \geq 3, \gamma > 0$. $\mathbb{P}(Z_R > \exp(-R^\gamma)) = \exp(-R^{f_d(\gamma)+o(1)})$ where*
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**Theorem**

*For all $d \geq 3$, $\gamma > 0$. $\mathbb{P}(Z_R > \exp(-R\gamma)) = \exp(-R^{f_d(\gamma)+o(1)})$ where for $d \geq 5$*

$$f_d(\gamma) = \begin{cases} 
1 + \frac{2-\gamma}{d-2} & 0 < \gamma \leq 2 \\
1 & \gamma \geq 2 
\end{cases}$$
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**Theorem**

*For all $d \geq 3, \gamma > 0$. $\mathbb{P}(Z_R > \exp(-R^{\gamma})) = \exp(-R^{f_d(\gamma) + o(1)})$ where for $d \geq 5$*

$$f_d(\gamma) = \begin{cases} 1 + \frac{2-\gamma}{d-2} & 0 < \gamma \leq 2 \\ 1 & \gamma \geq 2 \end{cases}$$

*and

$$f_3(\gamma) = \begin{cases} 3 - 2\gamma & 0 \leq \gamma \leq 1 \\ 1 & \gamma \geq 1 \end{cases}$$

$$f_4(\gamma) = \begin{cases} 1 + \frac{2-\gamma}{2} & 0 \leq \gamma \leq 4/3 \\ 4 - 2\gamma & 4/3 \leq \gamma \leq 3/2 \\ 1 & \gamma \geq 3/2 \end{cases}$$
Papers

- Gravitational allocation to Poisson Points. To Appear in Annals of Mathematics.
- Phase Transitions in Gravitational Allocation. In Preparation.
Potential energy function

- We introduce a stationary gravitational potential energy function $U(x)$ satisfying $F(x) = -\nabla U(x)$

\[
U(x) := \frac{1}{d-2} \lim_{T \to \infty} \left[ \frac{d^2 \kappa_d}{2} T^2 - \sum_{i : |z_i - x| < T} \frac{1}{|z_i - x|^{d-2}} \right]
\] (4)

This limit unfortunately converges only for $d \geq 5$ which presents some added complications for $d = 3, 4$. We shall not discuss these here.
Figure: The random planar potential for the Gaussian entire function (courtesy of Manjunath Krishnapur).
Liouville’s theorem

Liouville’s theorem: If \( \dot{\Gamma}_x(t) = F(\Gamma_x(t)) \) and \( \text{div}(F) = \alpha \) everywhere, then for every (bounded measurable) set \( A \subseteq \mathbb{R}^d \)

\[
\text{Vol}(\Gamma_A(t)) = \text{Vol}(A)e^{\alpha t}
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Liouville’s theorem

Liouville’s theorem: If $\dot{\Gamma}_x(t) = F(\Gamma_x(t))$ and $\text{div}(F) = \alpha$ everywhere, then for every (bounded measurable) set $A \subseteq \mathbb{R}^d$

$$\text{Vol}(\Gamma_A(t)) = \text{Vol}(A)e^{\alpha t}$$

In our case $\alpha = d \kappa_d$ at all points except stars. The theorem still holds so long as $A$ does not hit a star during its evolution up to time $t$. 
Having force of order $R^{1-\gamma}$ to the right and force to the outside on edges will cause a tentacle of length $R$ and mass $\exp(-R\gamma)$ to form. The proof is by considering the backward flow and using Liouville’s theorem (similar to Nazarov, Sodin and Volberg).
The Dense Galaxy Effect occurs when a region of space of radius $R$ (say, a ball or cylinder) has $cR^{d-\gamma}$ stars more than its expectation. Its mass is so large that it causes massive tentacles (mass $\exp(-R^\gamma)$) to be pulled into it from far away (distance $R$). This is the dominant effect for $d = 3$ and for $d = 4$ and $\frac{4}{3} \leq \gamma \leq \frac{3}{2}$. 
In high dimensions the dominant way in which long tentacles are formed is by “Wormholes”. Long thin tubes (length $R$, radius $R^{-\frac{2-\gamma}{d-2}}$) in space having stars on their boundary in rings of increasing intensity (density $R^{(2-\gamma)\frac{d-1}{d-2}}$ with respect to surface area). In dimension 4, a transition is made between the Dense Galaxy Effect and the Wormhole Effect when $\gamma$ goes below $\frac{4}{3}$. 