Gravitational allocation to Poisson points

Ron Peled

joint work with

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Dan Romik

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where $\mathrm{Vol}(\,\cdot\,)$ is Lebesgue measure in $\mathbb{R}^d$.

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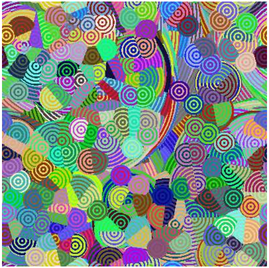
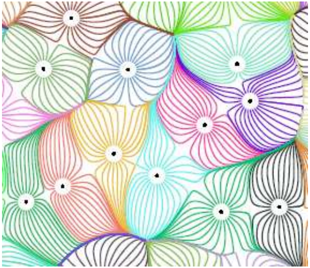
  where $\mathrm{Vol}(\cdot)$ is Lebesgue measure in $\mathbb{R}^d$.
- For $z \in \Xi$, we call $\psi^{-1}(z)$ the **cell allocated to** z.

Examples



(a)



(b)

Figure: (a) The two-dimensional stable marriage allocation for a Poisson process (picture due to Alexander E. Holroyd). (b) The gradient flow allocation (picture due to Manjunath Krishnapur).

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- One object of interest: The rate of decay of the tail $\mathbb{P}(X > R)$.

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- Questions about allocations are, in some sense, infinite volume analogues of the matching problem. Results on finite volume matchings can yield existence of good *randomized* equivariant allocations, but new techniques are required to construct *non-randomized* equivariant allocations.

Allocation to Poisson points: Existing results

- Using Talagrand's results, Holroyd & Peres constructed a randomized allocation for the Poisson process in dimension $d \geq 3$ with $\mathbb{P}(X > R) \leq C_d \exp(-c_d R^d)$, where X is the allocation diameter (their result is for $Y = |\psi_Z(0)|$, but a small variant of the proof works for the allocation diameter).

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- In the stable marriage allocation, a.s.
  1. all the cells are bounded and contain their owners,
  2. but not all are connected,
  3. and when Z is a Poisson point process the allocation diameter X satisfies $\mathbb{E}X^d = \infty$.

Picture of a stable marriage allocation

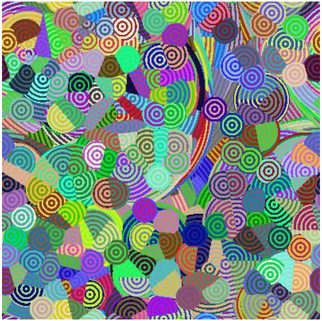

Figure: The 2-diml. stable marriage allocation for a Poisson process

Construction: Each point of the process grows a ball at unit rate and captures all the sites it reaches first, until it obtains volume 1.

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- Consider the random stationary vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ ("the force") defined by

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- First investigated in work of S. Chandrasekhar. Later work by Heath & Shepp.
- For $d \geq 3$ the force converges a.s. to a continuously differentiable vector field, off the stars.

Gravitational allocation

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- Define the **gravitational allocation rule**

$$\psi_{\mathcal{Z}}(x) = \begin{cases} z & x \in B(z) \text{ for } z \in \mathcal{Z}, \\ \infty & x \notin \bigcup_{z \in \mathcal{Z}} B(z). \end{cases}$$

Picture of gravitational allocation

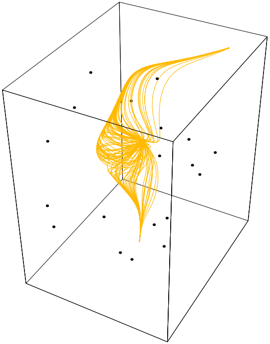

Figure: Simulation of a cell in 3-dimensional gravitational allocation

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- It is not hard to see that equality of expectations hold, then equality of the RV's is shown by controlling the variance of the difference.

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- Thus, by the divergence theorem,

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- Combining, we get $\mathrm{Vol}(B(z)) = 1$.

First result

Theorem

The mapping $\mathcal{Z} \to \psi_{\mathcal{Z}}$ is an allocation rule of Lebesgue measure to the Poisson point process $\mathcal{Z}$. Almost surely all the cells $\psi^{-1}(z)$ are bounded. The allocation diameter $X = diam(\psi^{-1}(\psi(0)))$ satisfies the following tail bounds: In dimensions 4 and higher, we have

$$\mathbb{P}(X > R) \le C_1 \exp\left[-c_2 R(\log R)^{\frac{d-2}{d}}\right] \tag{2}$$

for some constants $C_1, c_2 > 0$ (depending on the dimension d) and all positive R. In dimension 3, for any $\alpha > 0$ there exist constants $C_1, c_2 > 0$ (depending on α) such that for all $R > 0$ we have

$$\mathbb{P}(X > R) \le C_1 \exp\left[-c_2 \frac{R}{(\log R)^{\frac{4}{3}+\alpha}}\right]. \tag{3}$$

Characteristic exponents

In a sequel work which is in writing, we give a lower bound for the tail of the diameter, identifying the characteristic exponent in the probability,

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Also, for $Y = |\psi(0)|$, the typical allocation distance, we show

Theorem

For all $d \geq 3$,

$$\mathbb{P}(Y > R) = \exp(-R^{\beta_d + o(1)}) \text{ as } R \to \infty$$

where $\beta_3 = 1$ and $\beta_d = 1 + \frac{1}{d-1}$ for $d \geq 4$.

Characteristic exponents II

All the previous results are corollaries of the following theorem. Let Z_R be the volume of the cell of the origin after a ball of radius R around $\psi(0)$ was removed from it. Then

Theorem

For all $d \geq 3, \gamma > 0$. $\mathbb{P}(Z_R > \exp(-R^\gamma)) = \exp(-R^{f_d(\gamma)+o(1)})$ *where*

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and

$$f_3(\gamma) = \begin{cases} 3 - 2\gamma & 0 < \gamma \leq 1 \\ 1 & \gamma \geq 1 \end{cases} \qquad f_4(\gamma) = \begin{cases} 1 + \frac{2-\gamma}{2} & 0 < \gamma \leq 4/3 \\ 4 - 2\gamma & 4/3 \leq \gamma \leq 3/2 \\ 1 & \gamma \geq 3/2 \end{cases}$$

Papers

- Gravitational allocation to Poisson Points. To Appear in Annals of Mathematics.
- Phase Transitions in Gravitational Allocation. In Preparation.

Potential energy function

- We introduce a stationary gravitational potential energy function $U(x)$ satisfying $F(x) = -\nabla U(x)$

$$U(x) := \frac{1}{d-2} \lim_{T\to\infty} [\frac{d\kappa_d}{2} T^2 - \sum_{i\,:\,|z_i - x| < T} \frac{1}{|z_i - x|^{d-2}}] \qquad (4)$$

  This limit unfortunately converges only for $d \geq 5$ which presents some added complications for $d = 3, 4$. We shall not discuss these here.

Picture of planar potential

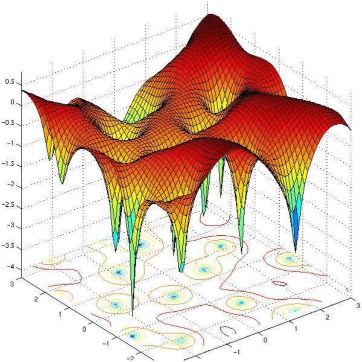

Figure: The random planar potential for the Gaussian entire function (courtesy of Manjunath Krishnapur).

Liouville's theorem

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$$\text{Vol}(\Gamma_A(t)) = \text{Vol}(A)e^{\alpha t}$$

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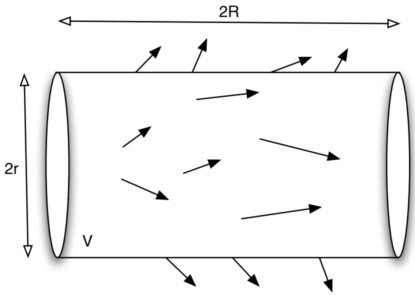
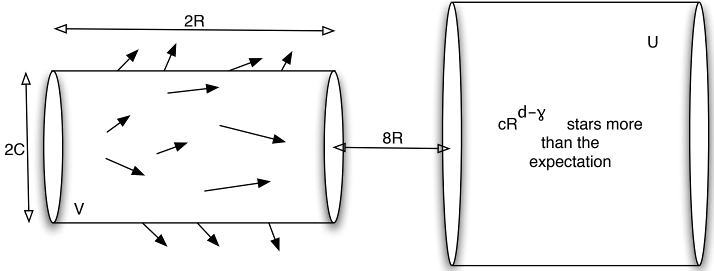
- In our case $\alpha = d\kappa_d$ at all points except stars. The theorem still holds so long as A does not hit a star during its evolution up to time t.

Tentacle creation



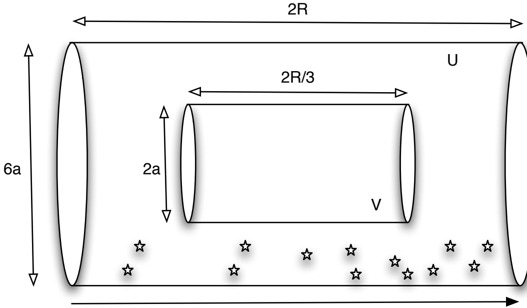
Having force of order $R^{1-\gamma}$ to the right and force to the outside on edges will cause a tentacle of length R and mass $\exp(-R^{\gamma})$ to form. The proof is by considering the backward flow and using Liouville's theorem (similar to Nazarov, Sodin and Volberg).

Dense Galaxy Effect



The Dense Galaxy Effect occurs when a region of space of radius R (say, a ball or cylinder) has $cR^{d-\gamma}$ stars more than its expectation. Its mass is so large that it causes massive tentacles (mass $\exp(-R^{\gamma})$) to be pulled into it from far away (distance R). This is the dominant effect for $d = 3$ and for $d = 4$ and $\frac{4}{3} \leq \gamma \leq \frac{3}{2}$.

Wormhole Effect



The density of stars on the boundary of the cylinder U increases linearly

In high dimensions the dominant way in which long tentacles are formed is by "Wormholes". Long thin tubes (length R, radius $R^{-\frac{2-\gamma}{d-2}}$) in space having stars on their boundary in rings of increasing intensity (density $R^{(2-\gamma)\frac{d-1}{d-2}}$ with respect to surface area). In dimension 4, a transition is made between the Dense Galaxy Effect and the Wormhole Effect when γ goes below $\frac{4}{3}$.