On K-wise Independent Distributions, Boolean Functions and Percolation

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joint work with

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and
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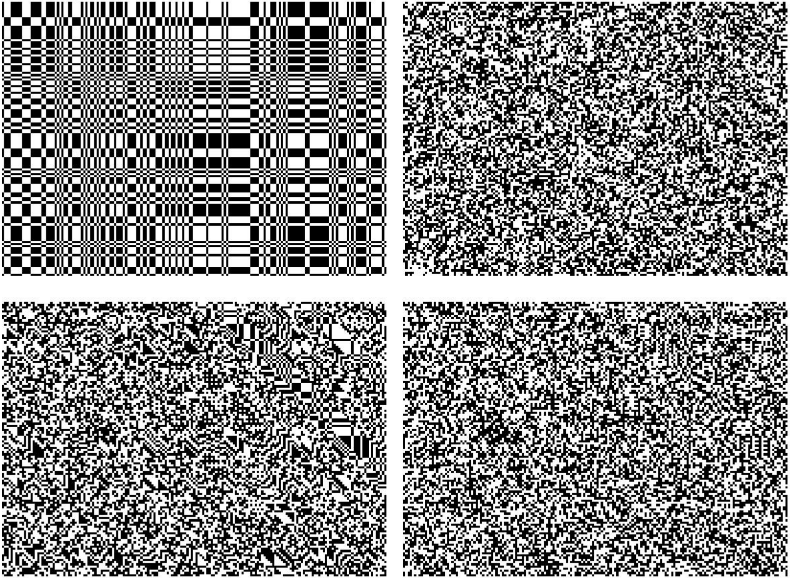
Percolation pictures



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- Define $\mathcal{A}(n, k, p)$ to be the set of all k-wise independent distributions $\mathbb{Q}$ on n bits with $\mathbb{Q}(X_i = 1) = p$ for all i.
- In this work we try to understand for a given function $f : \{0,1\}^n \to \{0,1\}$ the quantities

$$\max_{\mathbb{Q} \in \mathcal{A}(n,k,p)} \mathbb{Q}(f = 1) \qquad \text{and} \qquad \min_{\mathbb{Q} \in \mathcal{A}(n,k,p)} \mathbb{Q}(f = 1)$$

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- Related concept of almost k-wise independence is also very important. It's use was pioneered for derandomization purposes by Naor-Naor 90 and developed further by Alon-Goldreich-Håstad-Peralta 90, Azar-Motwani-Naor 90, Alon-Bruck-Naor-Naor-Roth 92, Even-Goldreich-Luby-Nisan-Velićković 92, Chari-Rohathi-Srinivisan 94, Alon-Goldreich-Mansour 03 and many others.

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- In this work we will concentrate only on (perfectly) *k*-wise independent distributions. Analogous questions can be asked for the almost *k*-wise independent case but we do not address these here.

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- Our questions are of a similar flavor, we ask, for a given boolean function f, how much independence is required for it to behave (about) the same on all k-wise independent inputs (including the completely independent one).

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  it is in principle enough to understand the extremal points of $\mathcal{A}(n, k, p)$. Unfortunately this appears to be very difficult and in the sequel we shall have to resort to ad-hoc methods for each function f we consider.
- For later reference, we identify the two extreme points of $\mathcal{A}(n, n-1, \frac{1}{2})$. XOR0 is the distribution on $(X_1, \ldots, X_n)$ having $\{X_i\}_{i=1}^{n-1}$ IID and $X_n \equiv \sum_{i=1}^{n-1} X_i$ mod 2, and XOR1 is the same with $X_n \equiv 1 + \sum_{i=1}^{n-1} X_i$ mod 2.

Parity

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  Does it necessarily behave the same on a k-wise independent input as on a fully independent input?
- No! In a very strong sense. For any $k < n$, under the XOR0 distribution the probability that parity returns 1 is 0 and under the XOR1 distribution the probability is 1.
- Hence to ensure that parity behaves normally one must take $k = n$!

Basic definitions

- Let $\varepsilon^f(k, p) := \max_{\mathbb{Q} \in \mathcal{A}(n,k,p)} \mathbb{Q}(f = 1) - \min_{\mathbb{Q} \in \mathcal{A}(n,k,p)} \mathbb{Q}(f = 1)$.

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- Define the δ-independence sensitivity of the function f at p to be

$$K(f,\delta,p) = \min(k \mid \varepsilon^f(k,p) \le \delta)$$

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- For simplicity, we define arbitrarily $K(f,p) := K(f,0.01,p)$ and call this the independence sensitivity of the function f at p.
- We will be mostly interested in monotone functions. We recall that a sequence $f_n : \{0,1\}^n \to \{0,1\}$ of monotone boolean functions has a sharp threshold at $p = p_c$ if ($\mathbb{P}_p$ is product distribution)

$$\lim_{n\to\infty} \mathbb{P}_p(f_n = 1) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases}$$

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- What about $p = \frac{1}{2}$? (can voters bias an election by using a voting scheme which is close to fully independent? can they do it if the scheme is only a little independent?)

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  The normal distribution is determined by its moments. Hence, if $k(n) \in \omega(1)$ and $\mathbb{Q}_n \in \mathcal{A}(n, k(n), 1/2)$ then $\overline{S_n} \to N(0,1)$ weakly. In particular, $\mathbb{Q}_n(\text{Maj}_n = 1) = \mathbb{Q}_n(\overline{S_n} > 0) \to 1/2$.

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  And when $\mathbb{Q}_0 \in \mathcal{A}(n, n-1, \frac{1}{2})$ *is the XOR0 distribution we have* $|\mathbb{Q}_0(\mathrm{Maj}_n = 1) - \frac{1}{2}| \geq \frac{1}{3\sqrt{n}}$.

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- The theorem shows that $K(\mathrm{Maj}_n, \frac{1}{2})$ is constant for all n (which tends to infinity with the arbitrary threshold 0.01). Voters cannot significantly bias the election even when only finite independence is required.

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  - The theorem shows that $K(\mathrm{Maj}_n, \frac{1}{2})$ is constant for all n (which tends to infinity with the arbitrary threshold 0.01). Voters cannot significantly bias the election even when only finite independence is required.
  - Upper bound (with worse constant) was known in coding theory (Sidel'nikov's theorem, see Macwilliams and Sloane). But our proof seems much simpler, it uses the theory of the classical moment problem (Akhiezer 65, Kreĭn-Nudel'man 77).

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  It is noise stable if for any ε there exists k such that the fraction of the Fourier mass below weight k is at least $1 - \varepsilon$.
- One may also define a quantitative version by setting $K_{\mathrm{NS}}(f)$ to be the minimal weight such that the fraction of Fourier mass above it is less than 0.01, say.

Iterated majority

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- It utilizes the duality of the problem to the problem of approximating the function by real polynomials.
- One of our main open questions is whether we can have $K_{\text{NS}}(f_n) = \omega(K(f_n, \frac{1}{2}))$.

Duality - approximation by polynomials

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- Similarly define $P_k^-(f)$ with $P(x) \leq f(x)$ on $\{0,1\}^n$.

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- Similarly define $P_k^-(f)$ with $P(x) \leq f(x)$ on $\{0,1\}^n$.
- By linear programming duality

$$\max_{\mathbb{Q}\in\mathcal{A}(n,k,p)} \mathbb{Q}(f=1) = \min_{P\in P_k^+(f)} \mathbb{E}_{\mathbb{P}_p} P(X_1,\ldots,X_n)$$

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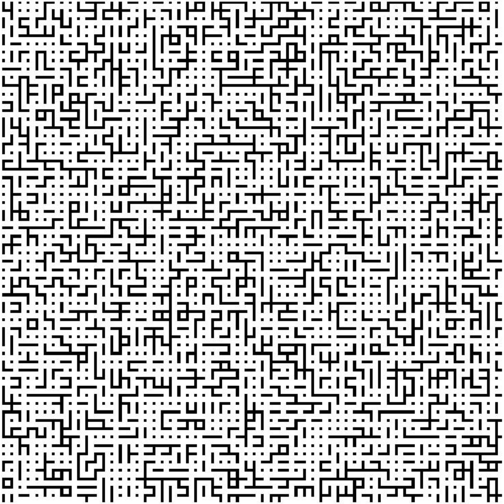
- Hence for f to behave the same under all k-wise independent distributions is equivalent to f having a "sandwich L_1" approximation by real polynomials of degree k.

Percolation with $p = \frac{1}{3}$



Percolation crossing

- Our main result is about the percolation crossing function. We consider percolation in a finite box in d dimensions ($d \geq 2$) with side length n. Consider the function f which says if there is a crossing from left to right. Recall that f has a sharp threshold at $0 < p_c < 1$.

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- What is $K(f, p)$? for example, how much independence is needed to have that for any $p > p_c$ the probability of crossing tends to 1 (with n) and for any $p < p_c$ the probability of crossing tends to 0?
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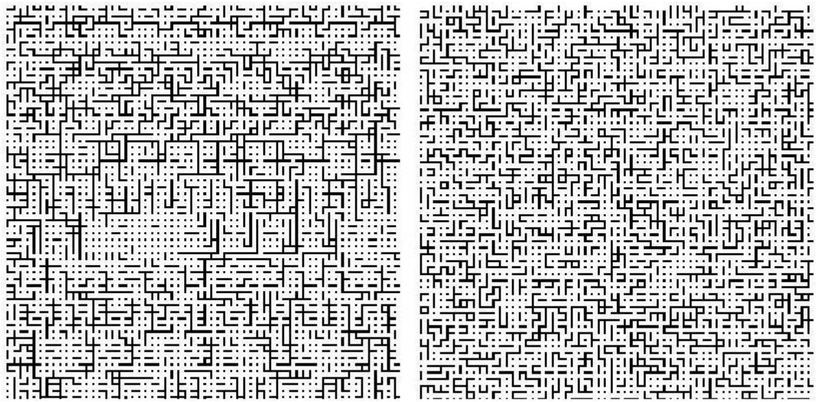
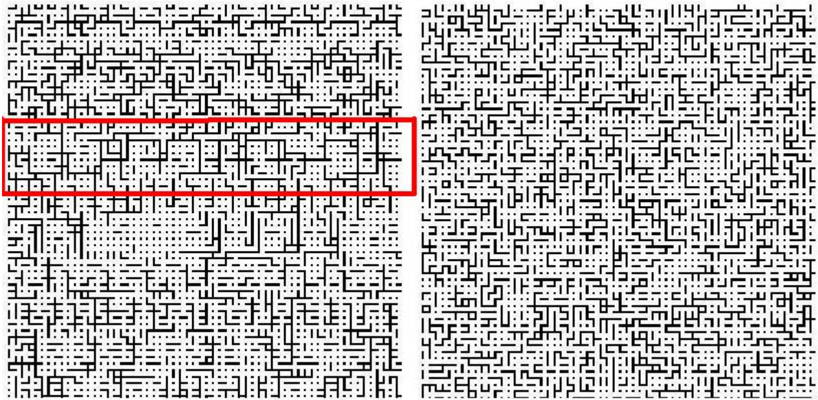
- We find that $(\log n)^{c_p/\sqrt{\log \log n}} \leq K(f, p) \leq C_p \log n$ asymptotically for $p \neq p_c$.
- When $d > 2$ we only know the upper bound for $p < p_c$.
- A similar result holds on the d-ary tree.
- This answers a question of Benjamini, Kozma and Romik.

2-wise percolation at $p = \frac{1}{3}$



2-wise percolation at $p = \frac{1}{3}$ with box



Orthogonal arrays

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- What is the analogue of this question for general k-wise independent distributions?
- Denote $L := |\Omega|$. If we suppose WLOG that Ω contains the all zeros string we have that the probability to sample this string is $\frac{1}{L}$.
- Hence one analogue of the question in our case is: How high can the probability of the all zeros string be?
- An upper bound on this quantity implies a lower bound on L, but it is more general since it also implies a bound on the atom at the all zeros string for any k-wise independent distribution.

Probability of all bits 1

- Another of our main results considers the following quantity

$$M(n, k, p) := \max_{\mathbb{Q} \in \mathcal{A}(n,k,p)} \mathbb{Q}(\text{All bits are 1})$$

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$$M(n,k,p) \le 2\sqrt{k}\left(\frac{kp}{2e(1-p)(n-\frac{k}{2})}\right)^{\frac{k}{2}} \le \left(\frac{Ckp}{(1-p)n}\right)^{\frac{k}{2}}$$

$$M(n,k,p) \le 2p^n \qquad \text{When } n(1-p) \le \frac{k}{2}$$

(second corollary uses a result of Jogdeo-Samuels 68).

Probability of all bits 1 contd.

- For $p = \frac{1}{2}$ the bound is a generalization of the bound of Alon-Babai-Itai 86 for the minimal size of an orthogonal array of bits. In this case we get an upper bound on the size of any atom of the distribution (by xoring a constant string).

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- For $p = \frac{1}{k}$ for an integer k, the bound is a generalization of the Rao bound 47 for the minimal size of an orthogonal array over $\mathbb{Z}_k$.

Proof of percolation theorem

- Our main lemma for the percolation result is inspired by the $(u \mid u+v)$ lemma of error-correcting codes (Macwilliams-Sloane 77). It allows to "amplify" independence.

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- Lemma

Fix $m \geq 1$. Let $X := (X_1, \ldots, X_n) \in \mathcal{A}^r(n,k)$. Let $X^i := (X^i_j)_{j=1}^n$ be m IID copies of X. Let also $Y := (Y_1, \ldots, Y_n) \in \mathcal{A}^r(n, 2k+1)$ be a vector independent of all the X's. Then the vector with the following coordinates

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  - We also have generalizations of this lemma which we do not present here.

Some open questions

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- Say anything non-trivial about the extremal points of $\mathcal{A}(n, k, p)$.
- What is K for percolation crossing in the plane at $p = p_c = \frac{1}{2}$?
- What is K at $p = \frac{1}{2}$ for iterated majority of height 3, for recursive majority of 3's?
- Can we have a boolean function whose Fourier spectrum is concentrated on high levels, but its K at $p = \frac{1}{2}$ is small? i.e. that $K_{\text{NS}}(f_n) = \omega(K(f_n, \frac{1}{2}))$.
- Conjecture of Linial-Nisan 90, about the independence sensitivity at $p = \frac{1}{2}$ for AC0 circuits.