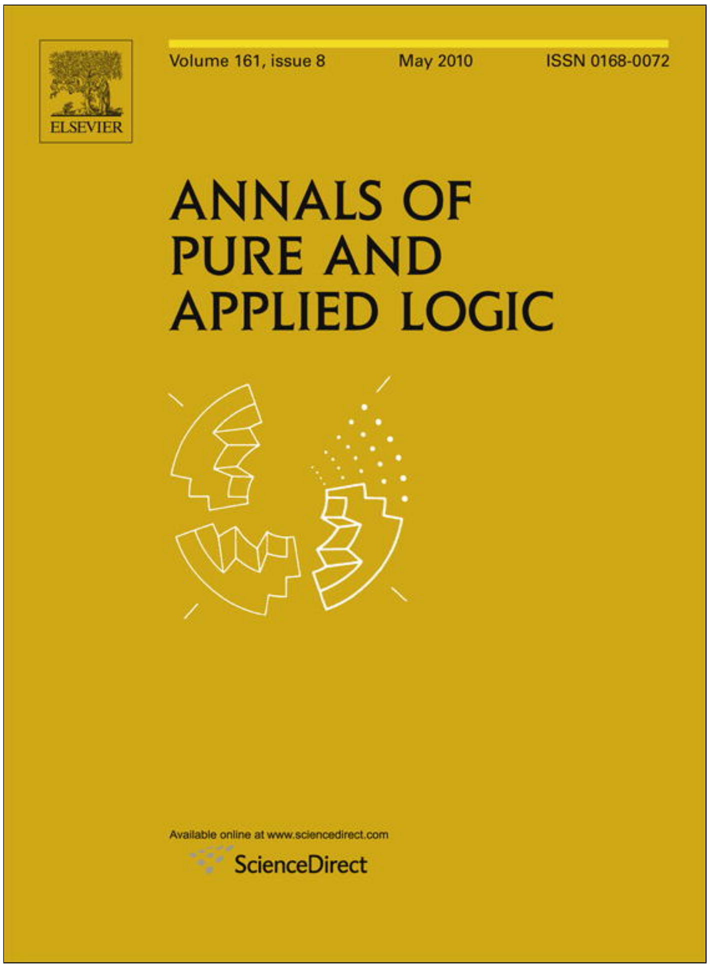
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Selection over classes of ordinals expanded by monadic predicates

Alexander Rabinovich*, Amit Shomrat

Sackler Faculty of Exact Sciences, Tel-Aviv University, 69978, Israel

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ABSTRACT

A monadic formula $\psi(Y)$ is a *selector* for a monadic formula $\varphi(Y)$ in a structure $\mathcal{M}$ if ψ defines in $\mathcal{M}$ a unique subset P of the domain and this P also satisfies φ in $\mathcal{M}$. If $\mathcal{C}$ is a class of structures and φ is a selector for ψ in every $\mathcal{M} \in \mathcal{C}$, we say that φ is a selector for φ over $\mathcal{C}$.

For a monadic formula $\varphi(X, Y)$ and ordinals $\alpha \leq \omega_1$ and $\delta < \omega^\omega$, we decide whether there exists a monadic formula $\psi(X, Y)$ such that for every $P \subseteq \alpha$ *of order-type smaller than* δ, $\psi(P, Y)$ selects $\varphi(P, Y)$ in $(\alpha, <)$. If so, we construct such a ψ.

We introduce a criterion for a class $\mathcal{C}$ of ordinals to have the property that *every* monadic formula φ has a selector over it. We deduce the existence of $S \subseteq \omega^\omega$ such that in the structure $(\omega^\omega, <, S)$ every formula has a selector.

Given a monadic sentence π and a monadic formula $\varphi(Y)$, we decide whether φ has a selector over the class of countable ordinals satisfying π, and if so, construct one for it.

1. Introduction

Definition 1.1 (*Uniformization*). Let $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$ be formulas and $\mathcal{C}$ a class of structures. We say that ψ *uniformizes* (or is a *uniformizer* for) φ *over* $\mathcal{C}$ iff for all $\mathcal{M} \in \mathcal{C}$:

1. $\mathcal{M} \models \forall\bar{X}\exists^{\leq 1}\bar{Y}\psi(\bar{X}, \bar{Y})$,
2. $\mathcal{M} \models \forall\bar{X}\forall\bar{Y}(\psi(\bar{X}, \bar{Y}) \to \varphi(\bar{X}, \bar{Y}))$, and
3. $\mathcal{M} \models \forall\bar{X}\big(\exists\bar{Y}\varphi(\bar{X}, \bar{Y}) \to \exists\bar{Y}\psi(\bar{X}, \bar{Y})\big)$.

Here $\bar{X}$, $\bar{Y}$ are tuples of distinct variables and "$\exists^{\leq 1}\bar{Y}$ ..." stands for "there exists at most one ...". The class $\mathcal{C}$ is said to have the *uniformization property* iff every formula φ has a uniformizer ψ over $\mathcal{C}$.

If $\mathcal{C} = \{\mathcal{M}\}$ consists of a single structure, we speak of uniformization *in* $\mathcal{M}$ rather than *over* $\mathcal{C}$.

In [5], Lifsches and Shelah characterize all trees having the uniformization property with respect to formulas of the *second-order monadic logic of order* (MLO). This logic extends first-order logic by allowing quantification over *subsets* of the domain. The binary relation symbol '$<$' is its only non-logical constant. In this paper, we assume that '$<$' is interpreted as a linear order of the domain. Thus, our structures are chains (or chains expanded by finitely many subsets of the domain). Note that we also assume that the free-variables in a formula are of second order, i.e. they range over subsets of the domain.[1]

Lifsches and Shelah show in particular that an ordinal α has the uniformization property iff $\alpha < \omega^\omega$. So, already in $(\omega^\omega, <)$ there are formulas lacking a uniformizer. This naturally leads to the following algorithmic problem:

* Corresponding author.
E-mail addresses: rabinoa@post.tau.ac.il (A. Rabinovich), shomrata@post.tau.ac.il (A. Shomrat).

[1] If the variables $\bar{Y}$ in φ of the previous definition were individual variables, ranging over elements of the domain, then the problem of constructing a uniformizer for φ would become trivial in an ordinal α. Indeed, for any tuple $\bar{P}$ of subsets of α, one could choose the lexicographically minimal tuple $\bar{Q}$ of *elements* of α such that $(\alpha, <) \models \varphi(\bar{P}, \bar{Q})$.

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Definition 1.2 (*Uniformization problem*). The *uniformization problem* over a class $\mathcal{C}$ is:

Input a formula $\varphi(\bar{X}, \bar{Y})$;
Task determine whether φ has a uniformizer over $\mathcal{C}$, and if so, construct one for it.

Since $\alpha < \omega^\omega$ has the uniformization property, for such α the uniformization problem consists only in computing a uniformizer for each formula. While Lifsches and Shelah took no interest in decidability and computability issues, from their proof one can extract an algorithm as follows:

Proposition 1.3 (*Uniformization Below ω^ω*). *There is an algorithm that, given $k \in \omega$ and $\varphi(\bar{X}, \bar{Y})$, computes a $\psi(\bar{X}, \bar{Y})$ that uniformizes φ in every ordinal smaller than ω^k.*

In the Appendix, we present a detailed proof of this proposition.

When we turn to $(\omega^\omega, <)$, things get trickier. So far, we have been unable to solve the uniformization problem in $(\omega^\omega, <)$. We succeeded, however, in solving some partial cases.

First, note that, strictly speaking, the input to the uniformization problem is not only a formula, but a formula plus a partition of its free-variables into *domain variables* $\bar{X}$ and *image variables* $\bar{Y}$.[2] *Selection* is the special case of uniformization where there are *no* domain variables:

Definition 1.4 (*Selection*). Let $\varphi(\bar{Y})$, $\psi(\bar{Y})$ be formulas and $\mathcal{C}$ a class of structures. We say that ψ *selects* (or, is a *selector* for) φ *over* $\mathcal{C}$ iff for every $\mathcal{M} \in \mathcal{C}$:

1. *either* both formulas are not satisfied in $\mathcal{M}$, *or*
2. ψ defines in $\mathcal{M}$ a unique tuple $\bar{P}$ and this $\bar{P}$ also satisfies φ in $\mathcal{M}$.

We say that $\mathcal{C}$ has the *selection property* iff every formula φ has a selector ψ over $\mathcal{C}$.

The *selection problem* over $\mathcal{C}$ is the restriction of the uniformization problem to formulas $\varphi(\bar{Y})$ given without domain variables.

Selection is treated in [9]. There we show, among other things:

Proposition 1.5. *For each $\alpha \geq \omega^\omega$, there are formulas lacking a selector in $(\alpha, <)$.*

On the other hand,

Proposition 1.6 (*Solvability of Selection in $\alpha \leq \omega_1$*). *There exists an algorithm that, given $\alpha \in [\omega^\omega, \omega_1]$ and a formula $\varphi(\bar{Y})$, decides whether φ has a selector in $(\alpha, <)$, and if so, constructs one for it.*[3]

The present paper continues the line of work began in [9] by tackling a problem which could be said to lie "in between" selection and the full uniformization problem.

The task of constructing a uniformizer is intuitively harder than that of constructing a selector in that a uniformizer must *respond* to a given tuple substituted for the domain variables $\bar{X}$ with an appropriate tuple to be substituted for the image variables $\bar{Y}$; it must (uniformly) answer a variety of challenges. In selection the $\bar{X}$ simply do not appear in the formula. Put more abstractly, their *variability* has been reduced to zero. A natural move therefore, when the $\bar{X}$ do appear in the formula, is to place various restrictions on the subsets of the domain substituted for them. One restriction which comes to mind is to consider formulas $\varphi(\bar{x}, \bar{Y})$ where the $\bar{x}$ are *individual* variables, i.e. range over elements of the domain. Once we show the solvability of the uniformization problem for such formulas, our next step may be to allow $\bar{X}$ to range only over *finite* subsets of the domain, or perhaps over sets of order-type ω, etc. These examples are generalized by the following definition.

Definition 1.7 (δ-*Uniformizer*). For ordinals δ and α, let $\mathcal{P}^{<\delta}(\alpha) := \{P \subseteq \alpha \mid \mathrm{otp}(P) < \delta\}$.[4]

Let $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$ be formulas. We say that ψ is a δ-uniformizer for φ in $(\alpha, <)$ iff clauses (1)-(3) of Definition 1.1 hold in $(\alpha, <)$ *when the $\bar{X}$ variables are restricted to range over members of $\mathcal{P}^{<\delta}(\alpha)$*.

The main result of this paper is:

Proposition 1.8 (*Solvability of Bounded Uniformization*). *There is an algorithm that, given ordinals $\alpha \in [\omega^\omega, \omega_1]$ and $\delta < \omega^\omega$ and a formula $\varphi(\bar{X}, \bar{Y})$, decides whether φ has a δ-uniformizer in $(\alpha, <)$, and if so, constructs one.*

---

2 In the few cases where we use letters other than X and Y, we shall state explicitly which variables are to be taken as domain variables and which as image variables.

3 Section 2.3 will clarify what we mean by the algorithm being given an ordinal as input. For now, the reader can read the proposition as stating merely that for each $\alpha \in [\omega^\omega, \omega_1]$, there is an algorithm which solves the selection problem in $(\alpha, <)$. A similar comment applies to Proposition 1.8.

4 As usual, $\mathrm{otp}(P)$ denotes the order-type of P.

Roughly speaking, our proof proceeds by reducing this problem to uniformization over the class of ordinals smaller than δ and to selection in $(\omega^\omega, <)$ (or in $(\omega_1, <)$ when $\alpha = \omega_1$). Proposition 1.3 tells us the former is solvable, while Proposition 1.6 handles the latter.

This paper expands on [9] in yet another direction. There we were mainly interested in whether a formula φ has a selector in $(\alpha, <)$ for a *particular* $\alpha \le \omega_1$. Here we ask whether it has a selector over a given *class* $\mathcal{C}$ of countable ordinals.

First, we prove the solvability of the selection problem over *definable* classes of countable ordinals. That is, given a sentence π and a formula $\varphi(\bar{Y})$, we decide whether φ has a selector over the class of countable ordinals satisfying π. When one exists, we construct it. Our proof reduces this problem to the bounded uniformization problem solved in Proposition 1.8.

Next, by Proposition 1.5, any class $\mathcal{C}$ of ordinals which has an $\alpha \ge \omega^\omega$ as a member lacks the selection property. On the other hand, by Proposition 1.3, any $\mathcal{C}$ bounded below ω^ω has it. It is therefore natural to ask whether there are unbounded $\mathcal{C} \subseteq \omega^\omega$ which have the selection property. We provide a simple necessary and sufficient condition for a class $\mathcal{C} \subseteq \omega^\omega$ to have this property, which implies the existence of unbounded $\mathcal{C}$'s having it.

In [9] we show that the formula stating "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$. On the other hand, given any formula $\varphi(\bar{Y})$, there is a $\psi(X, \bar{Y})$ such that if any unbounded ω-sequence $S \subseteq \omega^\omega$ is substituted for X, then $\psi(S, \bar{Y})$ selects φ in $(\omega^\omega, <)$. Thus, with an unbounded ω-sequence S as parameter, we can select every formula in $(\omega^\omega, <)$. This does not entail, however, that $(\omega^\omega, <, S)$ has the selection property, because the formulas φ for which ψ as above were constructed do not themselves refer to S. In fact, it is easy to find unbounded ω-sequences $S \subseteq \omega^\omega$ such that $(\omega^\omega, <, S)$ lacks the selection property. This lead us to ask:

Question 1.9 (*[9]*). Is there a finite tuple $\bar{P}$ of subsets of ω^ω such that $(\omega^\omega, <, \bar{P})$ has the selection property?

Using the existence of unbounded subclasses of ω^ω having the selection property, we are able here to provide an affirmative answer to this question. In fact, we shall show:

Proposition 1.10. *There are $P \subseteq \omega^\omega$ such that:*

(a) *$(\omega^\omega, <, P)$ has the selection property,*
(b) *the monadic theory of $(\omega^\omega, <, P)$ is decidable, and*
(c) *given a formula φ, we can compute a selector for φ in $(\omega^\omega, <, P)$.*[5]

The paper is organized as follows. In Section 2, we fix our notations and terminology. We also recall the basics of the 'composition method,' the main technical tool used in our proofs. Section 3 introduces an abstract framework for studying the selection problem over classes of chains (expanded by finitely many monadic predicates). Using this framework, we present a condition both necessary and sufficient for a formula φ to have a selector over a class $\mathcal{C}$ of chains, when $\mathcal{C}$ satisfies certain assumptions (see Lemmas 3.16 and 3.23). Section 4 shows that these assumptions apply in the case of bounded uniformization, which allows us to prove Proposition 1.8. In Section 5, we handle the selection problem over definable classes of countable ordinals, as explained above. Finally, Section 6 treats selection over classes $\mathcal{C} \subseteq \omega^\omega$, which are not necessarily definable and proves Proposition 1.10. As mentioned, the Appendix provides a proof of Proposition 1.3.

Note finally that, for the convenience of the reader, our treatment of selection over classes of countable ordinals was kept almost entirely independent from our proof of the solvability of bounded uniformization. There is only one point in Sections 5 and 6 where familiarity with either Section 3 or 4 is truly required, namely, in proving Proposition 1.10. There we make use of the Inheritance Lemma (3.12). To understand this lemma (proved in Section 3.2), the reader must familiarize himself/herself with the notations and definitions of Section 3.1. The technically more complicated conditions developed in Sections 3.3 and 3.4 are unnecessary. In any case, the reader who is willing to accept this one application of Lemma 3.12 on faith, can read the last two sections of this paper directly after Section 2.

2. Preliminaries and background

2.1. Notation and terminology

We use n, k, l, m, p, q for natural numbers, $\alpha, \beta, \gamma, \delta, \zeta, \mu$ for ordinals. Our ordinals are von Neumann ordinals: an ordinal is *identical* with the set of all ordinals below it. In particular, $0 = \emptyset, 1 = \{0\} = \{\emptyset\}, 2 = \{0, 1\}$, etc. $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. ω_1 is the first uncountable ordinal. We write $\alpha + \beta, \alpha\beta, \alpha^\beta$ for the sum, multiplication and exponentiation, respectively, of ordinals α and β.

For sets A and B, we denote by ${}^B A$ the set of all functions from B into A.

We use the expressions "*chain*" and "*linear order*" interchangeably. We use standard notation for sub-intervals of a chain: if $(A, <)$ is a chain and $b < a$ are in A, we write $(b, a) := \{c \in A \mid b < c < a\}$, $[b, a) := (b, a) \cup \{b\}$, etc.

We use the symbol '$\cong$' for isomorphism.

[5] Here φ *is* allowed to refer to P.

2.2. The Monadic Logic of Order (MLO)

2.2.1. Syntax

The vocabulary of MLO consists of *monadic* second-order variables X_i ($i \in \omega$) and binary relation symbols '$<$' and '$\subseteq$'. Atomic formulas take the form $X_i < X_j$ or $X_i \subseteq X_j$. All other formulas are built up from these by means of the usual Boolean connectives and second-order quantifiers $\exists X_i, \forall X_i$. The *quantifier depth* of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

We use upper-case letters $X, Y, \ldots$ to denote variables; with an overline, $\bar{X}, \bar{Y}$, etc. to denote finite tuples of variables (always assumed distinct).

2.2.2. Semantics

A *structure* is a tuple $\mathcal{M} := (A, <^{\mathcal{M}}, \bar{P}^{\mathcal{M}})$ where: A is a non-empty set, $<^{\mathcal{M}}$ is a binary relation on A, and $\bar{P}^{\mathcal{M}} := \langle P_0^{\mathcal{M}}, \ldots, P_{l-1}^{\mathcal{M}} \rangle$ is a *finite* tuple of sub*sets* of A.

If $\lg(\bar{P}^{\mathcal{M}}) = l$, we call $\mathcal{M}$ an *l-structure*. If $<^{\mathcal{M}}$ linearly orders A, we call $\mathcal{M}$ an *l-chain*. When the specific l is unimportant, we simply say that $\mathcal{M}$ is a *labeled chain*.

Suppose $\mathcal{M}$ is an l-structure and φ a formula with free-variables among $X_0, \ldots, X_{l-1}$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as follows: $\mathcal{M} \models X_i \subseteq X_j$ iff $P_i^{\mathcal{M}} \subseteq P_j^{\mathcal{M}}$, and $\mathcal{M} \models X_i < X_j$ iff there are $b <^{\mathcal{M}} a$ in A with $P_i^{\mathcal{M}} = \{b\}$, $P_j^{\mathcal{M}} = \{a\}$. The Boolean connectives are handled as usual and quantifiers range over subsets of A.[6]

Let $\mathcal{M}$ be an l-structure. The *monadic theory* of $\mathcal{M}$, $\mathrm{MTh}(\mathcal{M})$, is the set of all formulas with free-variables among $X_0, \ldots, X_{l-1}$ satisfied by $\mathcal{M}$.

Henceforth, we omit the superscript '$^{\mathcal{M}}$' in '$<^{\mathcal{M}}$' and '$\bar{P}^{\mathcal{M}}$'. We often write $(A, <) \models \varphi(\bar{P})$, meaning $(A, <, \bar{P}) \models \varphi$. Note also the following notations endemic to this paper:

Notation 2.1. Let $\mathcal{M} := (A, <, \bar{P})$ be a structure, $\bar{Q}$ a finite tuple of subsets of A. The *expansion* of $\mathcal{M}$ by $\bar{Q}$ is $\mathcal{M}^\frown \bar{Q} := (A, <, \bar{P}, \bar{Q})$, where we write '$\bar{P}, \bar{Q}$' meaning the tuple obtained by concatenating $\bar{P}$ and $\bar{Q}$.

Definition 2.2. Let $l_1, l_2 \in \omega$, $\mathcal{M}$ an l_1-structure, $\varphi(\bar{X}, \bar{Y})$ a formula with $\lg(\bar{X}) = l_1$ and $\lg(\bar{Y}) = l_2$.

1. The *relation defined by* φ *in* $\mathcal{M}$ is
   $\mathcal{D}(\varphi, \mathcal{M}) := \{\bar{Q} \in {}^{l_2}\mathcal{P}(\mathrm{dom}(\mathcal{M})) \mid \mathcal{M}^\frown \bar{Q} \models \varphi\}$.
2. Of every $\bar{Q} \in \mathcal{D}(\varphi, \mathcal{M})$, we say that it *satisfies* φ in $\mathcal{M}$.
3. When $\mathcal{D}(\varphi, \mathcal{M})$ is a singleton $\{\bar{Q}\}$, we say that φ *defines* $\bar{Q}$ in $\mathcal{M}$ and that $\bar{Q}$ is *definable* in $\mathcal{M}$.

2.2.3. First-order variables

We occasionally wish to have a variable range only over *elements* of the domain (equivalently, singleton subsets thereof). Since it is easy to write a formula $\mathrm{Sing}(X)$ stating "X is a singleton set", this can be achieved without formally adding first-order variables to our vocabulary. To distinguish them, we denote by lower-case letters x, y, etc., those variables ranging only over elements of the domain. For instance, "Let $\varphi(x, Y)$ be a formula ..." implies that φ has the form "$\mathrm{Sing}(X) \wedge \varphi'(X, Y)$" for some formula φ'.

2.2.4. Restriction

Notation 2.3. Let $\mathcal{M} := (A, <, \bar{P})$ be a structure and $\varnothing \neq D \subseteq A$. The *restriction* of $\mathcal{M}$ to D is the structure $\mathcal{M}_{\restriction D} := (D, <, \bar{P} \check{\cap} D)$ where $\bar{P} \check{\cap} D := \langle P_0 \cap D, \ldots, P_{l-1} \cap D \rangle$.

Lemma 2.4 (*Restriction*). *Let $\varphi(\bar{Y})$ be a formula, U a variable not appearing in φ. We can compute a formula $\varphi_{\restriction U}(\bar{Y}, U)$ such that for every* $\lg(\bar{Y})$*-structure $\mathcal{M}$ and every non-empty subset D of its domain,*

$$\mathcal{M}^\frown D \models \varphi_{\restriction U}(\bar{Y}, U) \quad \textit{iff} \quad \mathcal{M}_{\restriction D} \models \varphi(\bar{Y}).$$

That is, $\varphi_{\restriction U}$ holds in $\mathcal{M}$ with U interpreted as D iff φ holds in the restriction of $\mathcal{M}$ to D.

When this is the case, we say that φ holds in $\mathcal{M}$ restricted *to D.*

We are mostly interested in the case where $\mathcal{M}$ is a labeled chain and D is an interval $[b, a)$ for some $b < a$ in $\mathcal{M}$.

2.3. The monadic theory of countable ordinals

Büchi (for instance [2]) has shown that there is a *finite* amount of data concerning any ordinal $\leq \omega_1$ which determines its monadic theory:

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6 Our definition is equivalent to the standard definition of MLO which allows both first-order and second-order variables over the signature '$<$'.

Theorem 2.5. *Let $\alpha \in [1, \omega_1]$. Write $\alpha = \omega^\omega \beta + \zeta$ where $\zeta < \omega^\omega$ (this can be done in a unique way). Then the monadic theory of $(\alpha, <)$ is determined by:*

1. *whether α is countable or $\alpha = \omega_1$,*
2. *whether $\alpha < \omega^\omega$, and*
3. *ζ.*

We can associate with every $\alpha \leq \omega_1$ a finite *code* which holds the data required in the previous theorem. This is clear with respect to (1) and (2). As for (3), if $\zeta \neq 0$, write

$$\zeta = \sum_{i \leq n} \omega^{n-i} \cdot a_{n-i}, \quad \text{where } n, a_i \in \omega \text{ for } i \leq n \text{ and } a_n \neq 0$$

(this, too, can be done in a unique way), and let the sequence $\langle a_n, \dots, a_0 \rangle$ encode ζ. The following is then implicit in [2]:

Theorem 2.6 (*Monadic Decidability Theorem*). *There is an algorithm that, given a sentence φ and the* code *of an $\alpha \in [1, \omega_1]$, determines whether $(\alpha, <) \models \varphi$.*

Comment. In this paper, whenever we say that an algorithm is "given an ordinal …" or "returns an ordinal …", we mean the *code* of the ordinal. This holds in particular for Propositions 1.6 and 1.8 (which fulfils the promise made in footnote 3).

Finally, note that the monadic theory of a structure $\mathcal{M}$ "knows" which formulas uniformize which others in a structure $\mathcal{M}$.

Definition 2.7 (*Uniformization Axiom*). For formulas $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$, the (ψ, φ)-*uniformization axiom*, denoted uni-ax (ψ, φ), is the conjunction of the sentences appearing in the definition of uniformization (Definition 1.1).

When discussing the special case of selection, we write sel-ax(ψ, φ) instead of uni-ax(ψ, φ).

Clearly, ψ uniformizes φ in $\mathcal{M}$ iff $\mathcal{M} \models$ uni-ax(ψ, φ).

2.4. Elements of the composition method

Our proofs make use of the technique known as the composition method.[7] To fix notations and to aid the reader not familiar with this technique, we briefly review those definitions and results that we require. A more detailed presentation can be found in [11] or [4], for instance.

2.4.1. Hintikka formulas and n-types

Notation 2.8. Let $n, l \in \omega$. Denote by $\mathfrak{Form}_{n,l}$ the set of formulas of quantifier depth $\leq n$ and with free-variables among $X_0, \dots, X_{l-1}$.

Definition 2.9. Let $n, l \in \omega$ and $\mathcal{M}, \mathcal{N}$ be l-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are *n-equivalent*, denoted $\mathcal{M} \equiv^n \mathcal{N}$, iff for every $\varphi \in \mathfrak{Form}_{n,l}$, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \omega$ and $l > 0$, the set $\mathfrak{Form}_{n,l}$ is infinite. However, it contains only finitely many semantically distinct formulas. So, there are finitely many $\equiv^n$-classes of l-structures. In fact, we can compute "representatives" for these classes:

Lemma 2.10 (*Hintikka Lemma*). *For $n, l \in \omega$, we can compute a* finite *$H_{n,l} \subseteq \mathfrak{Form}_{n,l}$ such that:*

(a) *For every l-structure $\mathcal{M}$, there is a* unique *$\tau \in H_{n,l}$ such that $\mathcal{M} \models \tau$.*
(b) *If $\tau \in H_{n,l}$ and $\varphi \in \mathfrak{Form}_{n,l}$, then either $\tau \models \varphi$ or $\tau \models \neg\varphi$. Furthermore, there is an algorithm that, given such τ and φ, decides which of these two possibilities holds.*

*Any member of $H_{n,l}$ we call an (n, l)-*Hintikka formula.

Definition 2.11 (*n-Type*). For $n, l \in \omega$ and $\mathcal{M}$ an l-structure, we denote by type$^n(\mathcal{M})$ the unique member of $H_{n,l}$ satisfied by $\mathcal{M}$ and call it the *n-type* of $\mathcal{M}$.

Thus, type$^n(\mathcal{M})$ determines (effectively) which formulas of quantifier depth $\leq n$ are satisfied by $\mathcal{M}$.

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7 Originating in [3], and adapted and ingeniously applied to the case of MLO in [10].

2.4.2. The ordered sum of labeled chains

We occasionally make use of the following notation.

Notation 2.12. Let $l \in \omega$ and $\{\bar{P}^\alpha \mid \alpha \in I\}$ a family of l-tuples of sets. For each $\alpha \in I$, let $\bar{P}^\alpha := \langle P_0^\alpha, \ldots, P_{l-1}^\alpha \rangle$. Then the $\check{\cup}$-*union* of $\{\bar{P}^\alpha \mid \alpha \in I\}$ is:

$$\check{\bigcup_{\alpha \in I}} \bar{P}^\alpha := \left\langle \bigcup_{\alpha\in I} P_0^\alpha, \ldots, \bigcup_{\alpha \in I} P_{l-1}^\alpha \right\rangle.$$

From a family of labeled chains which is itself indexed by a linear order, there is a natural way of obtaining a new labeled chain:

Definition 2.13. Let $l \in \omega$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a linear order, and $\mathfrak{S} := \langle \mathcal{M}_\alpha \mid \alpha \in I\rangle$ a sequence of l-chains. Write $\mathcal{M}_\alpha := (A_\alpha, <^\alpha, \bar{P}^\alpha)$ and assume $A_\alpha \cap A_\beta = \varnothing$ whenever $\alpha \neq \beta$ are in I. The *ordered sum* of $\mathfrak{S}$ w.r.t. $\mathcal{I}$ is the l-chain

$$\sum_{\mathcal{I}} \mathfrak{S} := \left(\bigcup_{\alpha \in I} A_\alpha, <^{\mathcal{I},\mathfrak{S}}, \check{\bigcup_{\alpha\in I}} \bar{P}^\alpha \right),$$

where:

if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathcal{I},\mathfrak{S}} a$ iff $\beta < \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathcal{M}_\alpha$ are not disjoint, replace them with isomorphic l-chains that have disjoint domains, and proceed as before.

The next proposition says that taking ordered sums preserves n-equivalence.

Proposition 2.14. *Let $n, l \in \omega$. Assume:*

1. *$(I, <)$ is a linear order,*
2. *$\langle \mathcal{M}_\alpha^0 \mid \alpha \in I\rangle$ and $\langle \mathcal{M}_\alpha^1 \mid \alpha \in I\rangle$ are sequences of l-chains, and*
3. *for every $\alpha \in I$, $\mathcal{M}_\alpha^0 \equiv^n \mathcal{M}_\alpha^1$.*

Then $\sum_{\alpha\in I} \mathcal{M}_\alpha^0 \equiv^n \sum_{\alpha \in I} \mathcal{M}_\alpha^1$.

2.4.3. The composition theorem

Notation 2.15. Let I and H be sets. An H-*partition* of I is a sequence $\bar{\mathfrak{B}} := \langle \mathfrak{B}_\tau \mid \tau \in H\rangle$ of disjoint sets such that $\bigcup_{\tau \in H} \mathfrak{B}_\tau = I$.

Proposition 2.14 justifies the following definition.

Definition 2.16. Let $(I, <)$ be a chain, $n, l \in \omega$ and $\bar{\mathfrak{B}}$ an $H_{n,l}$-partition of I. For every $\alpha \in I$ denote by τ_α the unique $\tau \in H_{n,l}$ such that $\alpha \in \mathfrak{B}_\tau$. Fix $\varphi \in \mathfrak{Form}_{n,l}$.

We say that $\bar{\mathfrak{B}}$ *induces* φ w.r.t. $(I, <)$ iff whenever $\langle \mathcal{M}_\alpha \mid \alpha \in I\rangle$ is a sequence of l-chains such that $\mathrm{type}^n(\mathcal{M}_\alpha) = \tau_\alpha$ for each $\alpha \in I$, we have

$$\mathrm{type}^n\left(\sum_{\alpha \in I} \mathcal{M}_\alpha\right) \models \varphi.$$

The next fundamental result of Shelah's ([10]) says that we may define in $(I, <)$ the class of φ-inducing partitions.

Theorem 2.17 (*Composition Theorem*). *Let $n, l \in \omega$ and $\varphi \in \mathfrak{Form}_{n,l}$. We can compute a formula $\vartheta_{\varphi\,\mathrm{Ind}}(\bar{V})$ where $\bar{V} := \langle V_\tau \mid \tau \in H_{n,l}\rangle$ such that if $(I, <)$ is a chain and $\bar{\mathfrak{B}}$ an $H_{n,l}$-partition of I, then:*

$(I, <) \models \vartheta_{\varphi\,\mathrm{Ind}}(\bar{\mathfrak{B}})$ iff $\bar{\mathfrak{B}}$ induces φ w.r.t. $(I, <)$.

Finally, as a special case of inducement, Proposition 2.14 allows us also to define the *sum* of Hintikka formulas.

Definition 2.18. Let $n, l \in \omega$ and $\tau_0, \tau_1 \in H_{n,l}$. The *sum* of τ_0 and τ_1, denoted $\tau_0 + \tau_1$, is an element of $H_{n,l}$ such that whenever $\mathcal{M}_0, \mathcal{M}_1$ are l-structures with $\mathrm{type}^n(\mathcal{M}_i) = \tau_i$ for $i \in 2$, we have $\mathrm{type}^n(\mathcal{M}_0 + \mathcal{M}_1) = \tau_0 + \tau_1$.

Since the monadic theory of $(2, <)$ is decidable, the Composition Theorem yields:

Lemma 2.19. *$\lambda n, l \in \omega.\lambda \tau_0, \tau_1 \in H_{n,l}.\tau_0 + \tau_1$ is recursive.*

3. Conditions for selectability over classes of labeled chains

Definition 3.1. Let $(A, <)$ be a linear order. Call $S \subseteq A$ a *segment* of $(A, <)$ if $(b, a) \subseteq S$ whenever $b < a$ are in S.

Here we present the notion of a *split* class: a class $\mathcal{C}$ of labeled chains is split by a formula θ if θ defines in every $\mathcal{M} \in \mathcal{C}$ a partition of $\mathcal{M}$ into sub*segments*. We use this notion to prove the Inheritance Lemma (3.12), which provides a sufficient condition for $\mathcal{C}$ to have the selection property. This lemma is, however, too weak to be used in proving Proposition 1.8, and we therefore generalize it into the Sufficiency-of-Safety Lemma, where a sufficient condition for a specific formula φ to have a selector over $\mathcal{C}$ is given. Finally, in Section 3.4 we show that – under the appropriate assumptions on $\mathcal{C}$ – this last sufficient condition is also necessary for φ to have a selector.

3.1. Basic framework

3.1.1. Two stages of selection over split classes

Definition 3.2 (*Splitting*). Let $l_1 \in \omega$, $\mathcal{M}$ an l_1-chain with domain A, and $\theta(\bar{X}, x, y)$ a formula with $\lg(\bar{X}) \leq l_1$ and x and y first-order variables.[8]

1. We call θ a *splitting* of $\mathcal{M}$ iff θ defines in $\mathcal{M}$ an equivalence relation on the elements of A whose classes are segments of $\mathcal{M}$.
2. If θ splits $\mathcal{M}$, denote by $\sim_\theta^{\mathcal{M}}$ the equivalence relation defined by θ in $\mathcal{M}$, denote by $I_{\mathcal{M}/\theta}$ the set of $\sim_\theta^{\mathcal{M}}$-classes, and let $\mathcal{I}_{\mathcal{M}/\theta} := (I_{\mathcal{M}/\theta}, <)$ denote $I_{\mathcal{M}/\theta}$ ordered by representatives. Call $\mathcal{I}_{\mathcal{M}/\theta}$ the *indexing order* of $\mathcal{M}$ w.r.t. θ.
3. Let $\mathcal{C}$ be a class of labeled chains. We call θ a *splitting* of $\mathcal{C}$ iff θ splits every $\mathcal{M} \in \mathcal{C}$.

Throughout this section, fix $l_1 \in \omega$, a class $\mathcal{C}$ of l_1-chains and a splitting θ of $\mathcal{C}$. Note that $\mathcal{M} = \sum_{S \in I_{\mathcal{M}/\theta}} \mathcal{M}_{\restriction S}$ for every $\mathcal{M} \in \mathcal{C}$. Fix a formula $\varphi(\bar{X}, \bar{Y})$ with $\lg(\bar{X}) = l_1$. To decide whether φ has a selector over $\mathcal{C}$, one must decide whether it is possible to definably pick in every $\mathcal{M} \in \mathcal{C}$ a *unique* tuple $\bar{Q}$ such that $\mathcal{M}^\frown \bar{Q} \models \varphi$. Write

$$n := \mathrm{qd}(\varphi), \qquad l_2 := \lg(\bar{Y}), \qquad l := l_1 + l_2.$$

By Proposition 2.14, the n-types of the summands $(\mathcal{M}^\frown \bar{Q})_{\restriction S}$ (for $S \in I_{\mathcal{M}/\theta}$) determine whether $\mathcal{M}^\frown \bar{Q} \models \varphi$. Accordingly, we may try and break the task of selecting $\bar{Q}$ into two stages roughly as follows:

Partition the indexing order: choose an $H_{n,l}$*-partition* $\bar{\mathfrak{B}}^{\mathcal{M}} := \langle \mathfrak{B}_\tau^{\mathcal{M}} \mid \tau \in H_{n,l} \rangle$ of $I_{\mathcal{M}/\theta}$ which induces φ w.r.t. $\mathcal{I}_{\mathcal{M}/\theta}$ (recall Definition 2.16) and further satisfies:

(Coh) for every $S \in I_{\mathcal{M}/\theta}$, $\mathcal{M}_{\restriction S} \models \exists \bar{Y} \tau_S$,

where τ_S is the unique $\tau \in H_{n,l}$ such that $S \in \mathfrak{B}_\tau$.

Local selection: for each $S \in I_{\mathcal{M}/\theta}$, select a $\bar{Q}_S \in {}^{l_2}\mathcal{P}(S)$ such that $\mathrm{type}^n((\mathcal{M}_{\restriction S})^\frown \bar{Q}_S) = \tau_S$. Then $\bar{Q} := \check{\bigcup}_{S \in I_{\mathcal{M}/\theta}} \bar{Q}_S$ ought to do the trick. Thus, intuitively, $\bar{\mathfrak{B}}^{\mathcal{M}}$ instructs one which n-type to realize in each summand $\mathcal{M}_{\restriction S}$ so that, globally, one satisfies φ.

Note that had we not required that $\bar{\mathfrak{B}}^{\mathcal{M}}$ satisfy (Coh), there would be cases where we could not choose $\bar{Q}_S$ in compliance with it; for instance, suppose $l_1 = l_2 = 1$, $\mathcal{M} = (A, <, P)$, and $\tau_S(X, Y)$ implies "Y is a non-empty subset of X", but $P \cap S$ happens to be empty.

3.1.2. Type partitions of the indexing order

By the Composition Theorem, whether a partition $\bar{\mathfrak{B}}^{\mathcal{M}}$ of $I_{\mathcal{M}/\theta}$ induces φ is fully determined by the monadic theory of $(\mathcal{I}_{\mathcal{M}/\theta})^\frown \bar{\mathfrak{B}}^{\mathcal{M}}$. No reference to $\mathcal{M}$ itself is necessary. With regard to condition (Coh), things stand differently. There is, generally, no reason to assume that the monadic theory of $\mathcal{I}_{\mathcal{M}/\theta}$ "knows" whether a given $\tau \in H_{n,l}$ is satisfiable in $\mathcal{M}_{\restriction S}$ (where $S \in I_{\mathcal{M}/\theta}$). This may clearly involve the particular $\mathcal{M}$. Note, however, that $\exists \bar{Y} \tau_S \in \mathfrak{Form}_{n+l_2, l_1}$, so (Coh) is equivalent to the requirement that

(Coh') for each $S \in I_{\mathcal{M}/\theta}$, $\mathrm{type}^{n+l_2}(\mathcal{M}_{\restriction S}) \models \exists \bar{Y} \tau_S$.

This motivates the two following definitions.

Definition 3.3 (*Coherence*). Let I be a set, $n, l_1, l_2 \in \omega$, $\bar{\mathfrak{T}}$ an H_{n+l_2,l_1}-partition of I, and $\bar{\mathfrak{B}}$ an H_{n,l_1+l_2}-partition of I. We say that $\bar{\mathfrak{T}}$ and $\bar{\mathfrak{B}}$ are *coherent* iff for every $S \in I$, $\sigma \in H_{n+l_2,l_1}$ and $\tau \in H_{n,l_1+l_2}$,

$$\text{if } S \in \mathfrak{T}_\sigma \cap \mathfrak{B}_\tau, \text{ then } \sigma(\bar{X}) \models \exists \bar{Y} \tau(\bar{X}, \bar{Y}).$$

Definition 3.4 (*k-Type Partition*). Let $k, m \in \omega$, $\mathcal{M}$ an m-chain and θ a splitting of $\mathcal{M}$. For each $\tau \in H_{k,m}$, write $\mathfrak{T}_\tau(\mathcal{M}/\theta) := \{S \in I_{\mathcal{M}/\theta} \mid \mathrm{type}^k(\mathcal{M}_{\restriction S}) = \tau\}$. The *k-type partition* of $\mathcal{M}$ w.r.t. θ is $\mathrm{TyPart}^k(\mathcal{M}/\theta) := \langle \mathfrak{T}_\tau \mid \tau \in H_{k,m} \rangle$.

8 See the discussion in Section 2.2.3.

Thus, (Coh) is the requirement that $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ and $\bar{\mathfrak{B}}^{\mathcal{M}}$ be coherent. Note that in $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$, this is expressible by a formula:

Lemma 3.5 (*Coherence Lemma*). *For $n, l_1, l_2 \in \omega$, we can compute a formula $\vartheta_{\mathrm{Coh}}(\bar{U}, \bar{V})$, with $\bar{U}$ indexed by H_{n+l_2,l_1} and $\bar{V}$ by H_{n,l_1+l_2}, which in every chain $(I, <)$ states:*

"$\bar{U}$ and $\bar{V}$ are coherent partitions of the domain."

Proof. Compute $\mathrm{Coh}_{n,l_1,l_2} := \{(\sigma, \tau) \in H_{n+l_2,l_1} \times H_{n,l_1+l_2} \mid \sigma \models \exists \bar{Y}\tau\}$ using (b) of the Hintikka Lemma. Then the following does the job:

$$\vartheta_{\mathrm{Coh}}(\bar{U}, \bar{V}) := \forall x\Big(\bigvee_{(\sigma,\tau)\in \mathrm{Coh}_{n,l_1,l_2}} (x \in U_\sigma \cap V_\tau)\Big) \wedge \text{“}\bar{U} \text{ and } \bar{V} \text{ are partitions”}. \quad \square$$

Finally, the next observation follows immediately from the definitions of coherence and inducement:

Lemma 3.6. *Let $n, l_1, l_2 \in \omega$, $\mathcal{M}$ an l_1-chain, θ a splitting of $\mathcal{M}$, and $\bar{Q}$ an l_2-tuple of subsets of* $\mathrm{dom}(\mathcal{M})$. *Then:*

(a) $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ *and* $\mathrm{TyPart}^n(\mathcal{M}{}^\frown\bar{Q}/\theta)$ *are coherent.*
(b) *If $\varphi \in \mathfrak{Form}_{n,l_1+l_2}$ and $\bar{Q} \in \mathcal{D}(\varphi, \mathcal{M})$, then* $\mathrm{TyPart}^n(\mathcal{M}{}^\frown\bar{Q}/\theta)$ *induces φ w.r.t. $\mathcal{I}_{\mathcal{M}/\theta}$.*

3.1.3. Interpreting $\mathcal{I}_{\mathcal{M}/\theta}$ in $\mathcal{M}$

By what was just stated, the first stage of the selection scheme suggested above amounts to selecting in $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ a partition $\bar{\mathfrak{B}}^{\mathcal{M}}$ such that $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta){}^\frown\bar{\mathfrak{B}}^{\mathcal{M}}$ satisfies both $\vartheta_{\varphi\,\mathrm{Ind}}(\bar{V})$ and $\vartheta_{\mathrm{Coh}}(\bar{U}, \bar{V})$. But, we of course are looking for a selector for φ that would work in $\mathcal{M}$. To this end, we now show that $\mathcal{M}$ *interprets* $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$.

Definition 3.7. Let A be a set and $\sim$ an equivalence relation on A. We say that a subset $T \subseteq A$ *respects* $\sim$ iff T is the union of $\sim$-classes. A finite tuple $\langle T_\tau \mid \tau \in H\rangle$ of subsets of A is said to *respect* $\sim$ iff T_τ does for all $\tau \in H$.

Notation 3.8. Let A be a set and $\sim$ an equivalence relation on A. If $T \subseteq A$, let $(T/\sim)$ denote the set of $\sim$-classes of members of T, and if $\bar{T} := \langle T_\tau \mid \tau \in H\rangle$ is a finite tuple of subsets of A, let $(\bar{T}/\sim) := \langle (T_\tau/\sim) \mid \tau \in H\rangle$.

Lemma 3.9 (*Interpretation Lemma*). *Let $\theta(\bar{X}, x, y)$ be a formula. Set $l_1 := \mathrm{lg}(\bar{X})$. Given formulas $\chi(\bar{X}, \bar{W})$ and $\vartheta(\bar{W}, \bar{V})$, we can compute a formula $\vartheta^{\frown\chi}_{/\theta}(\bar{X}, \bar{V})$ with the following property:*

If $\mathcal{M}$ is an l_1-chain split by θ, $\chi(\bar{X}, \bar{W})$ defines in $\mathcal{M}$ a tuple $\bar{T}$, and $\bar{B} \in {}^{\mathrm{lg}(\bar{V})}\mathcal{P}(\mathrm{dom}(\mathcal{M}))$, then:

$\mathcal{M}{}^\frown\bar{B} \models \vartheta^{\frown\chi}_{/\theta}$ *iff $\bar{B}$ respects $\sim^{\mathcal{M}}_\theta$ and* $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown(\bar{T}/\sim^{\mathcal{M}}_\theta){}^\frown(\bar{B}/\sim^{\mathcal{M}}_\theta) \models \vartheta$.

Applying the lemma with $\bar{T} = \check{\bigcup}\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ interprets $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ in $\mathcal{M}$, as desired.

Proof. First, it is easy to write a $\vartheta_{\mathrm{resp}}(W)$ which says "W respects $\sim^{\mathcal{M}}_\theta$". Indeed, this is equivalent to W being a union of $\sim^{\mathcal{M}}_\theta$-classes. To ensure $\vartheta_{/\theta}$ is satisfied only by $\bar{T}$ which respect $\sim^{\mathcal{M}}_\theta$, we assume it has the form $\vartheta_\star \wedge \bigwedge_{i<\mathrm{lg}(\bar{W})} \vartheta_{\mathrm{resp}}(W_i)$. The definition of $\vartheta_\star$ itself proceeds by induction on ϑ. We leave to the reader the proof that $\vartheta_{/\theta}$ has the desired property.

$(\mathrm{Sing}(W))_\star$ says "W is a $\sim^{\mathcal{M}}_\theta$-class". $(W < W')_\star$ says "both W and W' are $\sim^{\mathcal{M}}_\theta$-classes and there exist $w \in W$ and $w' \in W'$ such that $w < w'$". If $\vartheta = W \subseteq W'$ or $\vartheta = \mathrm{Emp}(W)$, let $\vartheta_\star := \vartheta$. Finally, $(\vartheta \wedge \vartheta')_\star := \vartheta_\star \wedge \vartheta'_\star$, $(\neg\vartheta)_\star := \neg(\vartheta_\star)$ and $(\exists Z\vartheta)_\star := \exists Z(\vartheta_{\mathrm{resp}}(Z) \wedge \vartheta_\star)$. $\square$

Remark 3.10. Suppose $\bar{W} = \bar{U}, \bar{V}$ and $\vartheta_{\bar{T}}(\bar{X}, \bar{U})$ defines in $\mathcal{M}$ a tuple $\bar{T}$ respecting $\sim^{\mathcal{M}}_\theta$. Then the lemma tells us that $\exists\bar{U}\big(\vartheta_{\bar{T}}(\bar{X}, \bar{U}) \wedge \vartheta_{/\theta}(\bar{X}, \bar{U}, \bar{V})\big)$ defines in $\mathcal{M}$ the class of tuples $\bar{B}$ which respect $\sim^{\mathcal{M}}_\theta$ and such that $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown(\bar{T}/\sim^{\mathcal{M}}_\theta){}^\frown(\bar{B}/\sim^{\mathcal{M}}_\theta) \models \vartheta_{/\theta}$.

3.2. Inheritability of the selection property

Notation 3.11. Let $\mathcal{C}$ be a class of labeled chains and θ a splitting of $\mathcal{C}$. Define:
$\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart} := \{\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^k(\mathcal{M}/\theta) \mid \mathcal{M} \in \mathcal{C} \wedge k \in \omega\}$,
$\mathrm{Smd}_{\mathcal{C}/\theta} := \{\mathcal{M}_{\restriction S} \mid \mathcal{M} \in \mathcal{C} \wedge S \in I_{\mathcal{M}/\theta}\}$.

The following lemma is a natural generalization of Proposition 6.1 in [5].

Lemma 3.12 (*Inheritance Lemma*). *Let $\mathcal{C}$ be a class of chains and θ a splitting of $\mathcal{C}$.*
If $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart}$ *and* $\mathrm{Smd}_{\mathcal{C}/\theta}$ *have the selection property, then so does $\mathcal{C}$.*
If further, selectors are computable over $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart}$ *and* $\mathrm{Smd}_{\mathcal{C}/\theta}$, *then the same holds over $\mathcal{C}$.*

Proof. Assume $\mathcal{C}$ is a class of l_1-chains for some $l_1 \in \omega$. Let $\varphi(\bar{X}, \bar{Y})$ with $\lg(\bar{X}) = l_1$. Assuming solvability of selection over $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart}$ and over $\mathrm{Smd}_{\mathcal{C}/\theta}$, we present an algorithm for the construction of a selector ψ for φ over $\mathcal{C}$, together with a proof for the correctness of the construction. The proof will make it clear that even without the solvability assumption, *existence* of a selector follows.[9] Write $n := \mathrm{qd}(\varphi)$, $l_2 := \lg(\bar{Y})$, $l := l_1 + l_2$.
(1) Compute $\vartheta_{\varphi\,\mathrm{Ind}}(\bar{V})$ as in the Composition Theorem and $\vartheta_{\mathrm{Coh}}(\bar{U}, \bar{V})$ as in the Coherence Lemma (computed from n, l_1, l_2). Recall that $\bar{U}$ is indexed by H_{n+l_2,l_1} and $\bar{V}$ by $H_{n,l}$. Let $\vartheta_1(\bar{U}, \bar{V}) := \vartheta_{\varphi\,\mathrm{Ind}} \wedge \vartheta_{\mathrm{Coh}}$.
(2) Since $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart}$ has the selection property and since selectors are computable over it, we may compute a selector $\vartheta_1^{\mathrm{sel}}(\bar{U}, \bar{V})$ for ϑ_1 over this class.
(3) Write a formula $\chi(\bar{X}, \bar{U})$ which defines $\check{\bigcup}\,\mathrm{TyPart}^{n+l_2,l_1}(\mathcal{M}/\theta)$ in every $\mathcal{M} \in \mathcal{C}$. This formula says:

"for every $\sigma \in H_{n+l_2,l_1}$ and $\sim_\theta^{\mathcal{M}}$-class W, $W \subseteq U_\sigma$ iff $\sigma(\bar{X})$ holds restricted to W".

(4) Compute $(\vartheta_1^{\mathrm{sel}})^{\frown\chi}_{\diagup\theta}(\bar{X}, \bar{V})$ as in the Interpretation Lemma ($\bar{U}$ here is $\bar{W}$ there).

Fix $\mathcal{M} \in \mathcal{C}$ in which φ is satisfied. By Lemma 3.6, $\vartheta_1(\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta), \bar{V})$ is satisfied in $\mathcal{I}_{\mathcal{M}/\theta}$. Indeed, for any $\bar{Q} \in \mathcal{D}(\varphi, \mathcal{M})$, $\mathrm{TyPart}^n(\mathcal{M}^\frown\bar{Q}/\theta)$ satisfies it. Since $\vartheta_1^{\mathrm{sel}}$ selects ϑ_1 in $\mathcal{I}_{\mathcal{M}/\theta}{}^\frown\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$, there exists a *unique* $\bar{\mathfrak{B}}^{\mathcal{M}}$ such that $\mathcal{I}_{\mathcal{M}/\theta} \models \vartheta_1^{\mathrm{sel}}(\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta), \bar{\mathfrak{B}}^{\mathcal{M}})$. By construction of ϑ_1, this $\bar{\mathfrak{B}}^{\mathcal{M}}$ is coherent with $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$ and induces φ. Write $\bar{B}^{\mathcal{M}} := \check{\bigcup}\bar{\mathfrak{B}}^{\mathcal{M}}$. By the Interpretation Lemma, $\vartheta_{\bar{B}^{\mathcal{M}}}(\bar{X}, \bar{V}) := (\vartheta_1^{\mathrm{sel}})^{\frown\chi}_{\diagup\theta}(\bar{X}, \bar{V})$ defines $\bar{B}^{\mathcal{M}}$ in $\mathcal{M}$. This concludes the stage of partitioning the index order in the two-stage scheme of selection described in Section 3.1.1.
(5) For each $\tau \in H_{n,l}$, compute a selector Ψ_τ for τ over $\mathrm{Smd}_{\mathcal{C}/\theta}$. This can be done because $\mathrm{Smd}_{\mathcal{C}/\theta}$ has the selection property and selectors are computable over this class.
(6) Let $\psi'(\bar{X}, \bar{Y})$ say:

"For every $\bar{V}$ which satisfies $\vartheta_{\bar{B}^{\mathcal{M}}}$, every $\sim_\theta^{\mathcal{M}}$-class W, and every $\tau \in H_{n,l}$,
$\Psi_\tau(\bar{X}, \bar{Y})$ holds restricted to W iff $W \subseteq V_\tau$."

Finally, let $\psi := \psi' \wedge \exists\bar{Y}\varphi$. Let us show that ψ selects φ over $\mathcal{C}$.

Fix $\mathcal{M} \in \mathcal{C}$. If φ is not satisfied in $\mathcal{M}$, then because $\exists\bar{Y}\varphi$ is a conjunct of ψ, neither is ψ. Assume then that φ is satisfied. By construction of $\vartheta_{\bar{B}^{\mathcal{M}}}$, ψ actually reads:

"For every $\sim_\theta^{\mathcal{M}}$-class W and $\tau \in H_{n,l}$, $\Psi_\tau(\bar{X}, \bar{Y})$ holds restricted to W iff $W \subseteq B_\tau^{\mathcal{M}}$."

Fix $S \in I_{\mathcal{M}/\theta}$ and let $\tau_S \in H_{n,l}$ such that $S \subseteq B_{\tau_S}^{\mathcal{M}}$. Note that this is equivalent to $S \in \mathfrak{B}_{\tau_S}^{\mathcal{M}}$ where $\bar{\mathfrak{B}}^{\mathcal{M}}$ is as above. Since $\bar{\mathfrak{B}}^{\mathcal{M}}$ is coherent with $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$, τ_S is satisfied in $\mathcal{M}_{\restriction S}$. Since Ψ_{τ_S} selects τ_S in $\mathcal{M}_{\restriction S}$, there is a unique $\bar{Q}_S \in {}^{l_2}\mathcal{P}(S)$ satisfying Ψ_{τ_S} in $\mathcal{M}_{\restriction S}$. We have $\mathrm{type}^n(\mathcal{M}_{\restriction S}{}^\frown\bar{Q}_S) = \tau_S$, of course. Let $\bar{Q} := \check{\bigcup}_{S \in I_{\mathcal{M}/\theta}} \bar{Q}_S$. Then $\bar{Q}$ satisfies ψ in $\mathcal{M}$. Also, $\mathrm{type}^n(\mathcal{M}^\frown\bar{Q}) \models \varphi$, since $\bar{\mathfrak{B}}^{\mathcal{M}}$ induces φ. Finally, let $\bar{Q}' \in \mathcal{D}(\psi, \mathcal{M})$ and $S \in I_{\mathcal{M}/\theta}$. Then $\bar{Q}'\check{\cap}S$ satisfies Ψ_{τ_S} in $\mathcal{M}_{\restriction S}$. But, Ψ_{τ_S} *defines* $\bar{Q}_S$ in $\mathcal{M}_{\restriction S}$, i.e., $\bar{Q}'\check{\cap}S = \bar{Q}_S$ and $\bar{Q}' = \bar{Q}$ is the unique tuple which satisfies ψ in $\mathcal{M}$. Done. $\square$

If $\mathrm{Ind}_{\mathcal{C}/\theta}$ has the uniformization property, then $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\,\mathrm{TyPart}$ has the selection property. Hence, we obtain the following corollary:

Corollary 3.13. *Let $\mathcal{C}$ be a class of chains and θ a splitting of $\mathcal{C}$. Assume:*

1. $\mathrm{Ind}_{\mathcal{C}/\theta}$ *has the uniformization property and*
2. $\mathrm{Smd}_{\mathcal{C}/\theta}$ *has the selection property.*

Then $\mathcal{C}$ has the selection property.

If further, both the uniformization problem over $\mathrm{Ind}_{\mathcal{C}/\theta}$ *and the selection problem over* $\mathrm{Smd}_{\mathcal{C}/\theta}$ *are solvable, then so is the selection problem over $\mathcal{C}$.*

Comment. In [6], the selection and uniformization properties for classes of structures constructed by the Feferman–Vaught generalized product (introduced in [3]) were investigated. It was shown that if classes K_1 and K_2 have the selection (respectively, uniformization) property, then the generalized product of these classes has the selection (respectively, uniformization) property. A splitting of a chain provides a representation of the chain as an ordered sum of chains. The ordered sum of chains is an instance of the generalized sum construct [10,8]. There is a natural generalization of Corollary 3.13 to the tree sum of trees — another instance of the generalized sum construct. It is interesting to investigate what instances of the generalized sum inherit the uniformization and selection properties.

As mentioned in the introduction, Sections 3.3, 3.4 and 4 are all geared towards the proof of Proposition 1.8. The reader more interested in our treatment of selection over classes of countable ordinals, may proceed directly to Section 5 without loss of continuity.

9 Wherever we write "we can compute a formula" replace "there exists a formula", etc.

3.3. Sufficient condition for selectability when $\mathrm{Smd}_{\mathcal{C}/\theta}$ lacks the selection property

We now generalize the Inheritance Lemma by relaxing the assumption that $\mathrm{Smd}_{\mathcal{C}/\theta}$ has the selection property. We shall have a finite family $\mathfrak{D}$ of subclasses of $\mathrm{Smd}_{\mathcal{C}/\theta}$. Given $\mathcal{M} \in \mathcal{C}$, we shall attempt to select the partition $\bar{\mathfrak{B}}^{\mathcal{M}}$ of $I_{\mathcal{M}/\theta}$ so that whenever $S \in \mathfrak{B}^{\mathcal{M}}_{\tau}$ for some type τ, $\mathcal{M}_{|S}$ belongs to a subclass $\mathcal{S} \in \mathfrak{D}$ over which τ has a selector, say $\Psi^{\mathcal{S}}_{\tau}$. If this can be done in *each* relevant $\mathcal{M}$, then at stage (5) of the construction above, we could use $\Psi^{\mathcal{S}}_{\tau}$ instead of a selector over $\mathrm{Smd}_{\mathcal{C}/\theta}$ (which may not exist). The next two definitions formalize (and slightly refine) this idea.

Definition 3.14 (*Multi-partition*). Let $\mathcal{C}$ be a class of labeled chains and $\theta(\bar{X}, x, y)$ a splitting of $\mathcal{C}$.

1. A *multi-partition* (*m.p.* for short) $\mathfrak{F}$ for $\mathcal{C}/\theta$ is given by fixing, for each $\mathcal{M} \in \mathcal{C}$, a partition $\bar{\mathfrak{F}}^{\mathcal{M}} := \langle \mathfrak{F}^{\mathcal{M}}_d \mid d \in D \rangle$ of the set $I_{\mathcal{M}/\theta}$ of $\sim^{\mathcal{M}}_{\theta}$-classes where D is some fixed finite set.
2. For $d \in D$, let $\mathcal{S}_d := \{\mathcal{M}_{|S} \mid \mathcal{M} \in \mathcal{C} \wedge S \in \mathfrak{F}^{\mathcal{M}}_d\}$. Also, $\mathfrak{D} := \{\mathcal{S}_d \mid d \in D\}$. We call members of $\mathfrak{D}$ *summand subclasses* and $\mathfrak{D}$ the *subclass family* of $\mathfrak{F}$.[10]
3. A formula $\theta_{\mathfrak{F}}(\bar{X}, \bar{Z})$ with $\bar{Z}$ indexed by D is said to *define* $\mathfrak{F}$ iff in every $\mathcal{M} \in \mathcal{C}$, it defines $\check{\bigcup}\bar{\mathfrak{F}}^{\mathcal{M}} := \langle \bigcup \mathfrak{F}^{\mathcal{M}}_d \mid d \in D \rangle$.[11]
4. We write $\mathrm{Ind}_{\mathcal{C}/\theta} {}^\frown \mathrm{TyPart} {}^\frown \mathfrak{F} := \{\mathcal{I}_{\mathcal{M}/\theta} {}^\frown \mathrm{TyPart}^k(\mathcal{M}/\theta) {}^\frown \bar{\mathfrak{F}}^{\mathcal{M}} \mid \mathcal{M} \in \mathcal{C} \wedge k \in \omega\}$.

Agreement. $d \leftrightarrow \mathcal{S}_d$ is a natural 1-1 correspondence between D and $\mathfrak{D}$. In what follows, we shall therefore assume $D = \mathfrak{D}$. Note that, under this agreement, if $\mathcal{S} \in \mathfrak{D}$ and an $\sim^{\mathcal{M}}_{\theta}$-class S belongs to $\mathfrak{F}^{\mathcal{M}}_{\mathcal{S}}$, then $\mathcal{M}_{|S} \in \mathcal{S}$.

Definition 3.15 (*Safety*). Let $l_1 \in \omega$, $\mathcal{C}$ a class of l_1-chains, and θ a splitting of $\mathcal{C}$ and $\mathfrak{F}$ an m.p. for $\mathcal{C}/\theta$ with subclass family $\mathfrak{D}$. Let $n, l_2 \in \omega$ and $\varphi \in \mathfrak{Form}_{n,l_1+l_2}$. We say that $\mathfrak{F}$ is *safe* for φ iff for every $\mathcal{M} \in \mathcal{C}$ in which φ is satisfied, there exists an H_{n,l_1+l_2}-partition $\bar{\mathfrak{B}}^{\mathrm{sf}}$ of $I_{\mathcal{M}/\theta}$ which induces φ, is coherent with $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$, and satisfies, for every $\tau \in H_{n,l_1+l_2}$ and $S \in \mathfrak{B}^{\mathrm{sf}}_{\tau}$,

(Safe) if $\mathcal{S} \in \mathfrak{D}$ is such that $S \in \mathfrak{F}^{\mathcal{M}}_{\mathcal{S}}$, then τ is selectable over $\mathcal{S}$.

We want to show that if $\mathrm{Ind}_{\mathcal{C}/\theta} {}^\frown \mathrm{TyPart} {}^\frown \mathfrak{F}$ has the selection property and a safe-for-φ and definable m.p. $\mathfrak{F}$ for $\mathcal{C}/\theta$ exists, then φ is selectable over $\mathcal{C}$. Note that if $\mathrm{Smd}_{\mathcal{C}/\theta}$ has the selection property and $\mathfrak{D} = \{\mathrm{Smd}_{\mathcal{C}/\theta}\}$, then $\mathfrak{F}$ is safe for every φ (because (Safe) holds vacuously). Thus, this result generalizes the Inheritance Lemma. The next lemma proves it but also adds conditions under which a selector for φ is computable.

Lemma 3.16 (*Sufficiency-of-Safety Lemma*). *Let $\mathcal{C}$ be a class of labeled chains, φ a formula. Suppose there are:*

a splitting θ^{φ} of $\mathcal{C}$,
a formula $\theta^{\varphi}_{\mathfrak{F}}$ which defines an m.p. $\mathfrak{F}^{\varphi}$ for $\mathcal{C}/\theta^{\varphi}$ with subclass family $\mathfrak{D}^{\varphi}$ such that $\mathrm{Ind}_{\mathcal{C}/\theta^{\varphi}} {}^\frown \mathrm{TyPart} {}^\frown \mathfrak{F}^{\varphi}$ *has the selection property.*

Then there is a formula ψ with the following property:

if $\mathfrak{F}^{\varphi}$ is safe for φ, then ψ selects φ over $\mathcal{C}$.

Assume further that we can compute θ^{φ} and $\theta^{\varphi}_{\mathfrak{F}}$ from φ, and solve the following problems:

(Ind-sol) *Selection over* $\mathrm{Ind}_{\mathcal{C}/\theta^{\varphi}} {}^\frown \mathrm{TyPart} {}^\frown \mathfrak{F}^{\varphi}$.
($\mathcal{S}$-sol) *Assume $\varphi \in \mathfrak{Form}_{n,l}$. Given $\tau \in H_{n,l}$ and $\mathcal{S} \in \mathfrak{D}^{\varphi}$, decide whether τ has a selector over $\mathcal{S}$, and – if so – construct one for it.*[12]

Then, ψ can be computed from φ.

Proof. The proof is an easy generalization of the one given for the Inheritance Lemma and we only indicate the necessary changes to be made. Assume $\mathcal{C}$ is a class of l_1-chains for some $l_1 \in \omega$. Let $\varphi(\bar{X}, \bar{Y})$ be given with $\mathrm{lg}(\bar{X}) = l_1$. Set $n := \mathrm{qd}(\varphi)$, $l_2 := \mathrm{lg}(\bar{Y})$, and $l := l_1 + l_2$. Proceed as follows.
(0) Compute $\theta^{\varphi}(\bar{X}, x, y)$ and $\theta^{\varphi}_{\mathfrak{F}}(\bar{X}, \bar{Z})$ (with $\bar{Z} := \langle Z_{\mathcal{S}} \mid \mathcal{S} \in \mathfrak{D}^{\varphi} \rangle$).
(1) Let $\vartheta_{\varphi\,\mathrm{Ind}}(\bar{V})$ and $\vartheta_{\mathrm{Coh}}(\bar{U}, \bar{V})$ as in (1) of the proof of the Inheritance Lemma. By assumption ($\mathcal{S}$-sol), we can compute for every $\mathcal{S} \in \mathfrak{D}^{\varphi}$,

$$\mathrm{Sel}^{\mathcal{S}}_{n,l} := \{\tau \in H_{n,l} \mid \tau \text{ is selectable over } \mathcal{S}\}.$$

10 Since it will always be clear which m.p. $\mathfrak{F}$ is under discussion, we omit mention of $\mathfrak{F}$ in our notation for $\mathfrak{D}$, though the latter clearly depends on the former.

11 Note that for every $d \in D$, $\bigcup \mathfrak{F}^{\mathcal{M}}_d$ is the set of elements of $\mathrm{dom}(\mathcal{M})$ whose $\sim^{\mathcal{M}}_{\theta}$-class is in $\mathfrak{F}^{\mathcal{M}}_d$.

12 Note that for any particular φ, the problem is trivially solvable, since $H_{n,l}$ is finite. We mean, of course, that there is a *uniform in* φ algorithm solving this problem for every φ. It is important to stress that we do *not* assume here that the full selection problem over $\mathcal{S}$ is solvable.

Define formulas:

$$\vartheta_{Safe}(\bar{Z},\bar{V}) := \bigwedge_{\mathcal{S}\in\mathfrak{D}^{\varphi}}\left(\forall x\in Z_{\mathcal{S}}\left(\bigvee_{\tau\in \mathrm{Sel}^{\mathcal{S}}_{n,l}} x\in V_{\tau}\right)\right), \text{and}$$

$$\vartheta_1(\bar{U},\bar{Z},\bar{V}) := \vartheta_{\varphi\,\mathrm{Ind}}\wedge\vartheta_{\mathrm{Coh}}\wedge\vartheta_{Safe}.$$

The conjunct ϑ_{Safe} in ϑ_1 is meant to ensure we select a partition $\bar{\mathfrak{B}}^{\mathrm{sf}}$ of $I_{\mathcal{M}/\theta^{\varphi}}$ which satisfies condition (Safe).
(2) Compute a selector $\vartheta_1^{\mathrm{sel}}(\bar{U},\bar{Z},\bar{V})$ for ϑ_1 over $\mathrm{Ind}_{\mathcal{C}/\theta^{\varphi}}{}^\frown\,\mathrm{TyPart}^\frown\,\bar{\mathfrak{F}}^{\varphi}$.
(3) Compute a formula $\chi(\bar{X},\bar{U},\bar{Z})$ which defines in every $\mathcal{M}\in\mathcal{C}$ the concatenation of $\check{\bigcup}\,\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta^{\varphi})$ and $\check{\bigcup}\bar{\mathfrak{F}}^{\mathcal{M}}$.
(4) Compute a formula $(\vartheta_1^{\mathrm{sel}})^{\frown\chi}_{/\theta^{\varphi}}(\bar{X},\bar{V})$ as in the Interpretation Lemma, letting $\bar{W}$ there be the concatenation of $\bar{U}$ and $\bar{Z}$.

Let $\mathcal{M}\in\mathcal{C}$ where φ is satisfied. Assume $\bar{\mathfrak{F}}^{\varphi}$ is safe for φ. Then there exists a partition $\bar{\mathfrak{B}}^{\mathrm{sf}}$ of $I_{\mathcal{M}/\theta^{\varphi}}$ as in Definition 3.15. Then $\vartheta_1(\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta^{\varphi}),\bar{\mathfrak{F}}^{\mathcal{M}},\bar{\mathfrak{B}}^{\mathrm{sf}})$ holds in $\mathcal{I}_{\mathcal{M}/\theta^{\varphi}}$. In particular, $\vartheta_1(\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta^{\varphi}),\bar{\mathfrak{F}}^{\mathcal{M}},\bar{V})$ is satisfied in $\mathcal{I}_{\mathcal{M}/\theta^{\varphi}}$. By choice of χ and $\vartheta_1^{\mathrm{sel}}$, and by the Interpretation Lemma, $(\vartheta_1^{\mathrm{sel}})^{\frown\chi}_{/\theta^{\varphi}}(\bar{X},\bar{V})$ defines in $\mathcal{M}$ the $\check{}$-union of a *unique* partition $\bar{\mathfrak{B}}^{\mathrm{sf}}$ of $I_{\mathcal{M}/\theta^{\varphi}}$ which is coherent with $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta^{\varphi})$, induces φ, and satisfies (Safe).
(5) For every $\mathcal{S}\in\mathfrak{D}^{\varphi}$ and $\tau\in\mathrm{Sel}^{\mathcal{S}}_{n,l}$, compute a selector $\Psi^{\mathcal{S}}_{\tau}$ for τ over $\mathcal{S}$.
(6) Then the required $\psi(\bar{X},\bar{Y})$ is the conjunction of $\exists\bar{Y}\varphi$ and:

"For the unique $\bar{Z}$ which satisfies $\theta^{\varphi}_{\bar{\mathfrak{F}}}$ (i.e. $\check{\bigcup}\bar{\mathfrak{F}}^{\mathcal{M}}$) and the unique $\bar{V}$ which satisfies $(\vartheta_1^{\mathrm{sel}})^{\frown\chi}_{/\theta^{\varphi}}(\bar{X},\bar{V})$ (i.e. $\check{\bigcup}\bar{\mathfrak{B}}^{\mathrm{sf}}$), for every $\sim^{\theta}_{\mathcal{M}}$-class W, $\tau\in H_{n,l}$ and $\mathcal{S}\in\mathfrak{D}^{\varphi}$:
$\Psi^{\mathcal{S}}_{\tau}(\bar{X},\bar{Y})$ holds restricted to W iff $\big(W\subseteq V_{\tau}$ and $W\subseteq Z_{\mathcal{S}}\big)$." □

3.4. *Universal structures and the necessity of safety*

The main result of this subsection is the Necessity-of-Safety Lemma (3.23) which introduces an assumption concerning m.p'.s $\mathfrak{F}$ for $\mathcal{C}/\theta$ under which the fact that $\mathfrak{F}$ is safe for φ (recall Definition 3.15) is not only sufficient but also *necessary* for the existence of a selector for φ over $\mathcal{C}$. This condition is tailored to our needs, so it is doubtful whether the lemma enjoys great generality. Our purpose in stating it explicitly is to isolate the essential (and simple) idea driving our proof of Proposition 1.8 below.

3.4.1. *The segment lemma*

As a first stage in developing our necessary condition, we would like to relate the existence of a selector for φ over $\mathcal{C}$ to the selectability of the types actually appearing in $\mathrm{TyPart}^n(\mathcal{M}^\frown\bar{Q}/\theta)$ where $\mathcal{M}\in\mathcal{C}$ and $\bar{Q}\in\mathcal{D}(\varphi,\mathcal{M})$. To this end, we prove:

Lemma 3.17 (*Segment Lemma*). *Let $\mathcal{M}$ be a labeled chain and ψ a formula which defines a tuple $\bar{Q}$ in $\mathcal{M}$. Let S be a segment of $\mathcal{M}$. Then $\mathrm{type}^{\mathrm{qd}(\psi)}((\mathcal{M}^\frown\bar{Q})_{\restriction S})$ defines $\bar{Q}\check{\cap}S$ in $\mathcal{M}_{\restriction S}$.*

Proof. Set $n:=\mathrm{qd}(\psi)$ and let $\bar{Q}'$ satisfy $\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S})$ in $\mathcal{M}_{\restriction S}$. We must show that $\bar{Q}'=\bar{Q}\check{\cap}S$.

Write $S^-:=\{b\in\mathrm{dom}(\mathcal{M})\mid\forall a\in S.b<a\}$ and $S^+:=\mathrm{dom}(\mathcal{M})\setminus(S^-\cup S)$. Both are segments of $\mathcal{M}$. Assume S^- and S^+ are non-empty. Then

$$\mathrm{type}^n(\mathcal{M}^\frown\bar{Q})=\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S^-})+\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S})+\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S^+}).$$

By assumption, $\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S})=\mathrm{type}^n(\mathcal{M}_{\restriction S}{}^\frown\bar{Q}')$, so

$$(\mathcal{M}^\frown\bar{Q})_{\restriction S^-}+\mathcal{M}_{\restriction S}{}^\frown\bar{Q}'+(\mathcal{M}^\frown\bar{Q})_{\restriction S^+}\models\psi.$$

But this structure equals $\mathcal{M}^\frown(\bar{Q}\check{\cap}S^-\cup\bar{Q}'\cup\bar{Q}\check{\cap}S^+)$. Since ψ defines $\bar{Q}$ in $\mathcal{M}$, it follows $(\bar{Q}\check{\cap}S^-\cup\bar{Q}'\cup\bar{Q}\check{\cap}S^+)=\bar{Q}$, so $\bar{Q}'=\bar{Q}\check{\cap}S$.

Finally, if $S^-=\varnothing$ ignore the leftmost summand above and if $S^+=\varnothing$, ignore the rightmost summand. □

3.4.2. *Universal structures and fat classes*

Suppose that ψ is a selector for φ over $\mathcal{C}$, $\mathcal{M}\in\mathcal{C}$, and $\bar{Q}$ the unique element of $\psi\in\mathcal{D}(\psi,\mathcal{M})$. Then the Segment Lemma says that for every $S\in I_{\mathcal{M}/\theta}$, $\mathrm{type}^n((\mathcal{M}^\frown\bar{Q})_{\restriction S})$ is selectable in $\mathcal{M}_{\restriction S}$. But how does that help us to choose a partition $\bar{\mathfrak{F}}^{\mathcal{M}}$ of $I_{\mathcal{M}/\theta}$ to satisfy (Safe) of Definition 3.15? How are we to relate selectability in the particular summand $\mathcal{M}_{\restriction S}$ to selectability over a class $\mathcal{S}\subseteq\mathrm{Smd}_{\mathcal{C}/\theta}$? The next definition is a first step in answering this question.

Definition 3.18 (*Selection Universal Structures*). Let $k_1,k_2\in\omega$, $\mathcal{S}$ a class of structures and $\mathcal{M}$ a structure. We say that $\mathcal{M}$ is $(k_1|k_2)$-*selection universal* in $\mathcal{S}$ iff for every formula $\Phi(\bar{Y})$ with $\mathrm{qd}(\exists\bar{Y}\Phi)\le k_1$, if there exists a ρ with $\mathrm{qd}(\rho)\le k_2$ which selects Φ in $\mathcal{M}$, then Φ is selectable over $\mathcal{S}$.

Lemma 3.22 presents what is for us the paradigmatic example of a class where selection universal structures can be found. To prove it, we need a lemma and a proposition, which are important in themselves.

Lemma 3.19 ($\mathfrak{p}$-*Lemma*). *There is a recursive* $\mathfrak{p} : \omega \to \omega$ *such that for each* $n \in \omega$, *any two non-0 countable multiples of* $\omega^{\mathfrak{p}(n)}$ *are* n*-equivalent.*

Proof. This is a special case of Theorem 3.5(B) in [10]. □

The following is Corollary 4.9 of [9].

Proposition 3.20. *There is an algorithm that, given a* $\varphi(\bar{Y})$ *selectable in* $(\omega^{\omega}, <)$, *constructs a* ψ *which selects* φ *in* $\omega^{\mathfrak{p}(\mathrm{qd}(\exists\bar{Y}\varphi))+1}\beta$ *for every* $\beta \in \omega_1 \setminus 1$, *where* $\mathfrak{p}$ *is as in the* $\mathfrak{p}$*-Lemma.*

Definition 3.21 (*Fat Class*). Let $\mathcal{S}$ be a class of countable ordinals and $p \in \omega$.

We call $\mathcal{S}$ *fat* iff for every $N \in \omega$, there is a non-0 multiple of ω^N in $\mathcal{S}$.

If further, every $\alpha \in \mathcal{S}$ is a multiple of ω^p, we call $\mathcal{S}$ p*-fat*.

In stating and proving the next lemma, we confuse an ordinal α with the structure $(\alpha, <)$. We shall continue to do so occasionally.

Lemma 3.22. *Let* $k_1 \in \omega$ *and* $\mathcal{S} \subseteq \omega_1$ *a* $(\mathfrak{p}(k_1) + 1)$*-fat class. Then for every* $k_2 \in \omega$, *there are* $\alpha \in \mathcal{S}$ *such that* $(\alpha, <)$ *is* $(k_1|k_2)$*-selection universal in* $\mathcal{S}$.

Proof. Let $N := \max\{\mathrm{qd}(\text{sel-ax}(\rho(\bar{Y}), \Phi(\bar{Y}))) \mid \mathrm{qd}(\exists\bar{Y}\Phi) \le k_1 \wedge \mathrm{qd}(\rho) \le k_2\}$ and $p := \mathfrak{p}(N) + 1$, with $\mathfrak{p}$ as in the $\mathfrak{p}$-Lemma. We assume $p \ge \mathfrak{p}(k_1) + 1$. Since $\mathcal{S}$ is fat, we can pick an $\alpha^* \in \mathcal{S}$ which is a non-0 multiple of ω^p. We claim that α^* is $(k_1|k_2)$-selection universal in $\mathcal{S}$. Indeed, let $\Phi(\bar{Y})$ such that $\mathrm{qd}(\exists\bar{Y}\Phi) \le k_1$ and assume ρ with $\mathrm{qd}(\rho) \le k_2$ selects Φ in $(\alpha^*, <)$. Then $(\alpha^*, <) \models \text{sel-ax}(\rho, \Phi)$. By choice of N, $\mathrm{qd}(\text{sel-ax}(\rho, \Phi)) \le N$. Since α^* is a multiple of ω^p, the $\mathfrak{p}$-Lemma tells us that $(\alpha^*, <)$ is N-equivalent to $(\omega^{\omega}, <)$. Thus, $(\omega^{\omega}, <) \models \text{sel-ax}(\rho, \Phi)$, which means that ρ selects Φ in $(\omega^{\omega}, <)$. In particular, Φ is selectable in $(\omega^{\omega}, <)$. Since every $\alpha \in \mathcal{S}$ is a multiple of $\omega^{\mathfrak{p}(k_1)+1}$ (recall $p \ge \mathfrak{p}(k_1) + 1$), Proposition 3.20 tells us that Φ has a selector over $\mathcal{S}$, as was to be shown. □

3.4.3. The Necessity-of-Safety Lemma

Lemma 3.23 (*Necessity-of-Safety Lemma*). *Let* $n, l_1, l_2 \in \omega$, $\mathcal{C}$ *a class of* l_1*-chains,* θ *a splitting of* $\mathcal{C}$ *and* $\mathfrak{F}$ *an m.p. for* $\mathcal{C}/\theta$ *with subclass family* $\mathfrak{D}$. *Assume that for every* $k \in \omega$ *and* $\mathcal{M} \in \mathcal{C}$ *there is a sequence* $\mathfrak{S}' := \langle \mathcal{M}'_S \mid S \in I_{\mathcal{M}/\theta}\rangle$ *of* l_1*-chains such that* $\sum_{\mathcal{I}_{\mathcal{M}/\theta}} \mathfrak{S}' \in \mathcal{C}$ *and for every* $S \in I_{\mathcal{M}/\theta}$:

$\mathcal{M}'_S \equiv^{n+l_2} \mathcal{M}_{\restriction S}$, *and*

if $\mathcal{S} \in \mathfrak{D}$ *and* $S \in \mathfrak{F}^{\mathcal{M}}_{\mathcal{S}}$, *then* $\mathcal{M}'_S$ *is* $(n + l_1 + l_2|k)$*-selection universal in* $\mathcal{S}$.

Let $\varphi \in \mathfrak{Form}_{n,l_1+l_2}$. *If* φ *has a selector over* $\mathcal{C}$, *then* $\mathfrak{F}$ *is safe for* φ.

Proof. Let ψ select φ over $\mathcal{C}$. Fix $\mathcal{M} \in \mathcal{C}$ where φ is satisfied. Write $l := l_1 + l_2$. We shall find an $H_{n,l}$-partition $\bar{\mathfrak{B}}^{\mathrm{sf}}$ of $I_{\mathcal{M}/\theta}$ as in Definition 3.15.

Let $k := \mathrm{qd}(\psi)$. Pick a sequence $\mathfrak{S}' := \langle \mathcal{M}'_S \mid S \in I_{\mathcal{M}/\theta}\rangle$ as in the assumption with reference to this k. Let $\mathcal{M}' := \sum_{\mathcal{I}_{\mathcal{M}/\theta}} \mathfrak{S}'$. Then $\mathcal{M}' \in \mathcal{C}$. Since each of the summands in $\mathcal{M}$ is $(n + l_2)$-equivalent to the corresponding summand in $\mathcal{M}'$, it follows that $\mathcal{M} \equiv^{n+l_2} \mathcal{M}'$. Since φ is satisfied in $\mathcal{M}$, we have $\mathcal{M} \models \exists\bar{Y}\varphi$, and therefore, $\mathcal{M}' \models \exists\bar{Y}\varphi$, because $\mathrm{qd}(\exists\bar{Y}\varphi) = n + l_2$. Since ψ is a selector for φ in $\mathcal{M}'$, there is a (unique) $\bar{Q}' \in \mathcal{D}(\psi, \mathcal{M}')$. For each $S \in I_{\mathcal{M}/\theta}$, let $S' := \mathrm{dom}(\mathcal{M}'_S)$. We assume the S' are disjoint for distinct S. For each $\tau \in H_{n,l}$, let $\mathfrak{B}^{\mathrm{sf}}_{\tau} := \{S \in I_{\mathcal{M}/\theta} \mid (\mathcal{M}'^\frown\bar{Q}')_{\restriction S'} \models \tau\}$. Set $\bar{\mathfrak{B}}^{\mathrm{sf}} := \langle \mathfrak{B}^{\mathrm{sf}}_{\tau} \mid \tau \in H_{n,l}\rangle$. We claim $\bar{\mathfrak{B}}^{\mathrm{sf}}$ satisfies the requirements of Definition 3.15.

Fix $\tau \in H_{n,l}$ and $S \in \mathfrak{B}^{\mathrm{sf}}_{\tau}$. Let $\mathcal{S} \in \mathfrak{D}$ such that $S \in \mathfrak{F}^{\mathcal{M}}_{\mathcal{S}}$. We first show that τ is selectable over $\mathcal{S}$. By definition of $\mathfrak{B}^{\mathrm{sf}}_{\tau}$, $\tau = \mathrm{type}^n((\mathcal{M}'^\frown\bar{Q}')_{\restriction S'})$. Write $\rho := \mathrm{type}^k((\mathcal{M}'^\frown\bar{Q}')_{\restriction S'})$. Since ψ selects φ in $\mathcal{M}'$, the Segment Lemma (3.17) tells us that ρ selects τ in $\mathcal{M}'_{\restriction S'} = \mathcal{M}'_S$ (recall $\mathrm{qd}(\psi) = k$). But, by choice of $\mathfrak{S}'$, $\mathcal{M}'_S$ is $(n + l|k)$-selection universal in $\mathcal{S}$, so τ is selectable over $\mathcal{S}$. Next, $\bar{Q}' \cap S'$ is witness that $\mathcal{M}'_S \models \exists\bar{Y}\tau$. But, again, $\mathcal{M}'_S \equiv^{n+l_2} \mathcal{M}_{\restriction S}$, hence also $\mathcal{M}_{\restriction S} \models \exists\bar{Y}\tau$. Thus, $\bar{\mathfrak{B}}^{\mathrm{sf}}$ is coherent with $\mathrm{TyPart}^{n+l_2}(\mathcal{M}/\theta)$. Finally, $\mathcal{M}'^\frown\bar{Q}' \models \varphi$, so $\bar{\mathfrak{B}}^{\mathrm{sf}}$ induces φ. This concludes our proof. □

Comment. Since $\mathcal{M}' \in \mathcal{C}$, θ is also a splitting of $\mathcal{M}'$. Note, however, that in the proof above there is no need to assume that the domains of the $\mathcal{M}'_S$ coincide with $\sim^{\mathcal{M}'}_{\theta}$-classes. In fact, we do not even require the assumption that $\mathcal{M}'_S \in \mathrm{Smd}_{\mathcal{C}/\theta}$ (though this is the case in our application of the lemma).

4. The uniformization problem with bounded domain variables

Here we prove the solvability of bounded uniformization (Proposition 1.8). To apply the Sufficiency/Necessity-of-Safety Lemmas – which deal with selection – to our problem, which is one of uniformization, we note that by changing the class of structures, we may view uniformization as a case of selection. Define:

Notation 4.1. For $l_1 \in \omega$ and ordinals δ and α, let $\mathrm{Exp}^{<\delta}_{l_1}(\alpha) := \{(\alpha, <, \bar{P}) \mid \bar{P} \in {}^{l_1}\mathcal{P}^{<\delta}(\alpha)\}$ be the class of all expansions of $(\alpha, <)$ by l_1-tuples of sets of order-type $< \delta$.

Then the following observation is obvious.

Lemma 4.2. *Let $l_1 \in \omega$, $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$ formulas with $\lg(\bar{X}) = l_1$, and δ and α ordinals. Then ψ is a δ-uniformizer for φ in $(\alpha, <)$ iff ψ selects φ over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$.*

Thus, what we must show is:

There is a (uniform in α, δ and l_1) algorithm which solves the selection problem over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ for all $\alpha \in [\omega^\omega, \omega_1]$, $\delta < \omega^\omega$ and $l_1 \in \omega$.

As usual, some preparation is needed.

Notation 4.3. For $l_1 \in \omega$, let $\epsilon_{l_1} := \langle \underbrace{\varnothing, \ldots, \varnothing}_{l_1 \text{ times}} \rangle$ be the l_1-tuple of the empty sets. If $\mathcal{S}$ is a class of structures, write $\mathcal{S}^\frown \epsilon_{l_1} := \{\mathcal{M}^\frown \epsilon_{l_1} \mid \mathcal{M} \in \mathcal{S}\}$.

We leave the proof of the following lemma to the reader.

Lemma 4.4. *Let $l_1 \in \omega$ and $\mathcal{S}$ a class of 0-structures. For $\Phi(\bar{X}, \bar{Y})$ with $\lg(\bar{X}) = l_1$, write $\Phi^*(\bar{Y}) := \exists \bar{X}(\Phi \wedge \bigwedge_{i<l_1}(X_i = \varnothing))$. Then Φ is selectable over $\mathcal{S}^\frown \epsilon_{l_1}$ iff Φ^* is selectable over $\mathcal{S}$, and given a selector for Φ^* over $\mathcal{S}$, we can compute a selector for Φ over $\mathcal{S}^\frown \epsilon_{l_1}$.*

Definition 4.5 (*Part and Tail*). Let μ, α be ordinals with $\mu > 0$. Write $\alpha = \mu\beta + \zeta$ with $\zeta < \mu$ (this can be done in a unique way). We call $\mu\beta$ the *μ-part* of α and ζ its *μ-tail*.

Finally, the following lemma is an easy exercise in formalization.

Lemma 4.6 (*Definability Below ω^ω*). *For any $\alpha < \omega^\omega$, we can compute sentences $\theta^{\mathrm{def}}_\alpha$ and $\theta^{\mathrm{def}}_{<\alpha}$ such that for every ordinal $\beta > 0$:*

(a) $(\beta, <) \models \theta^\alpha_{\mathrm{def}}$ *iff* $\beta = \alpha$.
(b) $(\beta, <) \models \theta^{<\alpha}_{\mathrm{def}}$ *iff* $\beta < \alpha$.

4.1. Proof of Proposition 1.8

Fix $\alpha \in [\omega^\omega, \omega_1]$ and $\delta < \omega^\omega$. If $\delta = 0$, then $\mathrm{Exp}^0_{l_1}(\alpha) = \varnothing$ and there is nothing to show. On the other hand, $\mathrm{Exp}^1_{l_1}(\alpha) = \{(\alpha, <, \epsilon_{l_1})\}$. Thus, Lemma 4.4 reduces the case $\delta = 1$ to the selection problem in $(\alpha, <)$, which is solvable by Proposition 1.6. We may therefore assume that $\delta > 1$.

The overall structure of the proof is as follows. Given φ, we write a splitting θ^φ of $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ and define an m.p. $\mathfrak{F}^\varphi$ for $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)/\theta^\varphi$. Both are computable from φ. We show that the assumptions of the Sufficiency-of-Safety Lemma are satisfied w.r.t. these θ^φ and $\mathfrak{F}^\varphi$. The lemma therefore allows us to *compute* a ψ such that if $\mathfrak{F}^\varphi$ is safe for φ, then ψ selects φ over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$. Next, we show that the assumptions of the Necessity-of-Safety Lemma hold. It follows that if φ has *any* selector over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$, then $\mathfrak{F}^\varphi$ is safe for φ, hence, by what was just stated, the above ψ is a selector for φ over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$. Combined, this means that φ has a selector iff ψ is a selector. This condition turns out to be easy to decide, thus completing the proof.

Let $\varphi(\bar{X}, \bar{Y})$ be given. Set $n := \mathrm{qd}(\varphi)$, $l_1 := \lg(\bar{X})$, $l_2 := \lg(\bar{Y})$, $l := l_1 + l_2$. Note that if $l_1 = 0$, then $\mathrm{Exp}^{<\delta}_0(\alpha) = \{(\alpha, <)\}$ and our problem is selection in $(\alpha, <)$. We henceforth assume $l_1 > 0$.

Compute $p := \mathfrak{p}(n + l) + 1$ where $\mathfrak{p}$ is as in the $\mathfrak{p}$-Lemma (3.19). Let α' denote the ω^ω-part of α. Write a splitting $\theta = \theta^\varphi(\bar{X}, x, y)$ of $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ as follows. For every $\bar{P} \in {}^{l_1}\mathcal{P}^{<\delta}(\alpha)$ and $\gamma < \beta < \alpha$, $\gamma \sim^{(\alpha,<,\bar{P})}_\theta \beta$ iff:

either $\gamma, \beta \geq \alpha'$, *or*
$\gamma, \beta < \alpha'$ and, if S is the set of ordinals strictly between the ω^p-part of γ and the ω^p-part of β, then $S \cap \bar{P}_i = \varnothing$ for all $i < l_1$.

This can indeed be done. Indeed, for any $\beta < \alpha$, $\beta \geq \alpha'$ iff $\mathrm{otp}([\beta, \alpha)) \leq \alpha - \alpha'$. But $\alpha - \alpha' < \omega^\omega$ (by definition of an ω^ω-part). By definability below ω^ω (4.6), this condition is expressible by a formula. Further, the ω^p-part of β is definable as the least $\beta' \leq \beta$ such that $\mathrm{otp}([\beta', \beta]) < \omega^p$, which again by definability below ω^ω is expressible. Note that θ is computable from φ.

The reader will now easily verify that for every $\mathcal{M} \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$, we have:

1. $\mathcal{I}_{\mathcal{M}/\theta}$ is isomorphic to an ordinal $< \delta\omega$,[13]
2. if $\alpha' \neq \alpha$, then $[\alpha', \alpha) \in I_{\mathcal{M}/\theta}$,

and for every $S \in I_{\mathcal{M}/\theta} \setminus \{[\alpha', \alpha)\}$:

3. there are $\gamma < \beta \leq \alpha'$ such that $S = [\omega^p\gamma, \omega^p\beta)$ (in particular, otp(S) is a non-0 multiple of ω^p), and
4. if $\mathcal{M}_{\restriction S} \neq (S, <, \epsilon_{l_1})$, then otp$(S) = \omega^p$.

Next, there is a formula $\theta_{\mathfrak{F}} = \theta^{\varphi}_{\mathfrak{F}}$ which defines an m.p. $\mathfrak{F} = \mathfrak{F}^{\varphi}$ for $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)/\theta$ as follows. Given $\mathcal{M} \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$, the m.p. partitions $I_{\mathcal{M}/\theta}$ into subsets $\mathfrak{F}^{\mathcal{M}}_{tail}$, $\mathfrak{F}^{\mathcal{M}}_{\omega^p}$, $\mathfrak{F}^{\mathcal{M}}_{>\omega^p}$, and $\mathfrak{F}^{\mathcal{M}}_{\omega_1}$: If $\alpha' = \alpha$, $\mathfrak{F}^{\mathcal{M}}_{tail} := \varnothing$; otherwise $\mathfrak{F}^{\mathcal{M}}_{tail} := \{[\alpha', \alpha)\}$. $\mathfrak{F}^{\mathcal{M}}_{\omega^p}$ and $\mathfrak{F}^{\mathcal{M}}_{>\omega^p}$ divide among them the $\sim^{\mathcal{M}}_{\theta}$-classes which have *countable* order-type and whose members are $<\alpha'$: $\mathfrak{F}^{\mathcal{M}}_{\omega^p}$ consists of all such $\sim^{\mathcal{M}}_{\theta}$-classes whose order-type is ω^p and $\mathfrak{F}^{\mathcal{M}}_{>\omega^p}$ of those whose order-type is strictly greater than ω^p (by (3), these are the only two options). If α is countable, this already exhausts all $\sim^{\mathcal{M}}_{\theta}$-classes and $\mathfrak{F}^{\mathcal{M}}_{\omega_1}$ is left empty. If $\alpha = \omega_1$, there is a unique $\sim^{\mathcal{M}}_{\theta}$-class whose order-type is ω_1 (because if $\mathcal{M} = (\omega_1, <, \bar{P})$ then $\bigcup_{i<l_1} P_i$ is bounded in ω_1). This class is the unique member of $\mathfrak{F}^{\mathcal{M}}_{\omega_1}$.

Denote by $\mathcal{S}_{tail}$, $\mathcal{S}_{\omega^p}$, $\mathcal{S}_{>\omega^p}$, and $\mathcal{S}_{\omega_1}$ the corresponding summand subclasses. Then:

$\mathcal{S}_{tail} = \varnothing$ if $\alpha' = \alpha$; otherwise, $\mathcal{S}_{tail} = \mathrm{Exp}^{<\delta}_{l_1}([\alpha, \alpha'))$,

Up to isomorphism, $\mathcal{S}_{\omega^p}$ is $\mathrm{Exp}^{<\delta}_{l_1}(\omega^p)$.

$\mathcal{S}_{\omega_1} = \varnothing$, if α is countable; otherwise, its unique member (up to isomorphism) is $(\omega_1, <, \epsilon_{l_1})$.

Let $\Omega := \{\omega^p\beta \mid \beta \in [2, \alpha']\}$. We claim that $\mathcal{S}_{>\omega^p}$ equals $\Omega^\frown\epsilon_{l_1}$, up to isomorphism. Indeed, that every member of $\mathcal{S}_{>\omega^p}$ is isomorphic to a member of $\Omega^\frown\epsilon_{l_1}$ follows directly from (3) and (4). Conversely, let $\beta \in [2, \alpha']$. Then $\mathcal{M}_\beta := (\alpha, <, \{\omega^p\beta\}\cap\alpha', \epsilon_{l_1-1}) \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ (recall that we assume $l_1 > 0$ and $\delta > 1$). Furthermore, $[0, \omega^p\beta)$ is a $\sim^{M_\beta}_{\theta}$-class which belongs to $\mathfrak{F}^{\mathcal{M}_\beta}_{>\omega^p}$ and $\mathcal{M}_\beta$ restricted to this class is $(\omega^p\beta, <, \epsilon_{l_1})$.

The subclass family of $\mathfrak{F}$ is $\mathfrak{D} = \{\mathcal{S}_{tail}, \mathcal{S}_{\omega^p}, \mathcal{S}_{>\omega^p}, \mathcal{S}_{\omega_1}\}$.

It is easy to see that $\theta_{\mathfrak{F}}$ is computable from φ (as was θ). We show that all other assumptions of the Sufficiency-of-Safety Lemma hold.

First, by Proposition 1.3, $\{(\beta, <) \mid \beta \in (\delta\omega) \setminus 1\}$ has the uniformization property and uniformizers are computable over this class. By (1), this means that $\mathrm{Ind}_{\mathrm{Exp}^{<\delta}_{l_1}(\alpha)/\theta}{}^\frown \mathrm{TyPart}^\frown \mathfrak{F}$ has the selection property and selectors are computable over it.

By the same proposition, $(\omega^p, <)$ and $([\alpha', \alpha), <)$ (if $\alpha' \neq \alpha$) have the uniformization property and uniformizers are computable in them. Thus, the same holds for selection w.r.t. $\mathrm{Exp}^{<\delta}_{l_1}(\omega^p)$ which is, up to isomorphism, $\mathcal{S}_{\omega^p}$ and w.r.t. $\mathrm{Exp}^{<\delta}_{l_1}([\alpha, \alpha')) = \mathcal{S}_{tail}$. Also, $\mathcal{S}_{\omega_1}$ is either empty or $\{(\omega_1, <, \epsilon_{l_1})\}$. The selection problem over it is therefore solvable (in the latter case, by Proposition 1.6). It remains to be seen that ($\mathcal{S}$-Sol) holds for $\mathcal{S}_{>\omega^p}$.

Let $\tau \in H_{n,l}$ and τ^* as in Lemma 4.4. By the lemma, τ is selectable over $\mathcal{S}_{>\omega^p}$ iff τ^* is selectable over Ω. But, τ^* is selectable over Ω iff it selectable in $(\omega^\omega, <)$. Indeed, $\Rightarrow$ is immediate since, as just shown, $\omega^\omega \in \Omega$. The $\Leftarrow$ direction is Proposition 3.20 and uses the fact that every member of Ω is a multiple of ω^p. By Proposition 1.6, we may decide whether τ^* is selectable in $(\omega^\omega, <)$. When this is the case, we use Proposition 3.20 again to compute a selector for τ^* over Ω, which we then translate into a selector for τ over $\mathcal{S}_{>\omega^p}$. This shows that ($\mathcal{S}$-sol) holds for $\mathcal{S}_{>\omega^p}$.

We may therefore compute a formula ψ as in Sufficiency-of-Safety Lemma, namely, such that if $\mathfrak{F}^{\varphi}$ is safe for φ, then ψ selects φ over $\mathrm{Exp}^{<\delta}_{l_1}(\alpha)$.

Next, we prove that $\mathfrak{F}$ satisfies the assumptions of the Necessity-of-Safety Lemma.

Fix $k \in \omega$ and $\mathcal{M} \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ where φ is satisfied. Define $\mathfrak{S}' := \langle \mathcal{M}'_S \mid S \in I_{\mathcal{M}/\theta} \rangle$ as follows. Let $q := \mathfrak{p}(\max\{n, k\} + 2l)$. Fix $S \in I_{\mathcal{M}/\theta}$. Write $\alpha_S := \mathrm{otp}(S)$. If $S \notin \mathfrak{F}^{\mathcal{M}}_{>\omega^p}$, take $\mathcal{M}'_S := \mathcal{M}_{\restriction S}$; if $S \in \mathfrak{F}_{>\omega^p}$, let $\mathcal{M}'_S$ be isomorphic to $(\omega^q\alpha_S, <, \epsilon_{l_1})$. We assume that the $\mathcal{M}'_S$ have been chosen so that for distinct $S \in I_{\mathcal{M}/\theta}$, the domains of the $\mathcal{M}'_S$ are disjoint. We claim:

• $\mathcal{M}' := \sum_{\mathcal{I}_{\mathcal{M}/\theta}} \mathfrak{S}' \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$ up to isomorphism. To show this, denote by α'_S the order-type of $\mathcal{M}'_S$. Then, on the one hand, for every $S \in I_{\mathcal{M}/\theta}$, $\alpha'_S \geq \alpha_S$, so otp$(\mathcal{M}') \geq \alpha$. On the other hand, because α' is a multiple of ω^ω, if $\alpha_S < \alpha'$, then also $\alpha'_S < \alpha'$, and if $\alpha_S = \alpha'$, then $\alpha'_S = \omega^q \cdot \alpha' = \alpha'$. Finally, $\alpha_S = \alpha'$ can only hold for the *top* S in $\mathcal{I}_{\mathcal{M}/\theta} \setminus \{[\alpha', \alpha)\}$ (if one exists). All in all, this implies that $\sum_{S \in I_{\mathcal{M}/\theta} \setminus \{[\alpha, \alpha')} \alpha'_S = \alpha'$. If $\alpha' = \alpha$, this means otp$(\mathcal{M}') = \alpha$. Otherwise, we must also note the segment $\mathcal{M}_{\restriction[\alpha', \alpha)}$ has not been changed, so again otp$(\mathcal{M}') = \alpha$. Thus, there exists a (unique) $\bar{P}' \in {}^{l_1}\mathcal{P}(\alpha)$ such that $\mathcal{M}' \cong (\alpha, <, \bar{P}')$. Fix $i < l_1$. Since in our construction of $\mathcal{M}'$, only segments whose intersection with P_i is empty were changed, it is easy to show otp$(P_i) = $ otp(P'_i). (Briefly, for each $S \in \mathfrak{F}^{\mathcal{M}}_{\omega^p}$, there exists a (unique) isomorphism $g_S : \mathcal{M}_{\restriction S} \to \mathcal{M}'_S$. If we let $f : \mathcal{M}' \to (\alpha, <, \bar{P}')$ be the unique isomorphism, then $(f \circ \bigcup_{S \in \mathfrak{F}^{\mathcal{M}}_{\omega^p}} g_S)|_{P_i}$ is an isomorphism $(P_i, <) \to (P'_i, <)$). In particular, $P'_i \in \mathcal{P}^{<\delta}(\alpha)$, so $\mathcal{M}' \cong (\alpha, <, \bar{P}') \in \mathrm{Exp}^{<\delta}_{l_1}(\alpha)$.

Fix $S \in I_{\mathcal{M}/\theta}$. We claim further:

• $\mathcal{M}'_S \equiv^{n+l_2} \mathcal{M}_{\restriction S}$. Indeed, if $S \notin \mathfrak{F}^{\mathcal{M}}_{>\omega^p}$, there is nothing to show. Assume $S \in \mathfrak{F}^{\mathcal{M}}_{>\omega^p}$. Then $\mathcal{M}'_S \cong (\omega^q\alpha_S, <, \epsilon_{l_1})$, so for every $\sigma \in H_{n+l_2,l_1}$, $\mathcal{M}'_S \models \sigma$ iff $(\omega^q\alpha_S, <) \models \sigma^*$. Since both α_S and $\omega^q\alpha_S$ are countable multiples of ω^p, we have $(\omega^q\alpha_S, <) \models \sigma^*$ iff $(\alpha_S, <) \models \sigma^*$ iff $(\alpha_S, <, \epsilon_{l_1}) \models \sigma$. But, $(\alpha_S, <, \epsilon_{l_1}) \cong \mathcal{M}_{\restriction S}$ and our claim follows.

[13] A better bound can easily be obtained.

• If $\mathcal{S} \in \mathfrak{D}$ and $S \in \mathfrak{F}^{\mathcal{M}}_{\mathcal{S}}$, then $\mathcal{M}'_S$ is $(n+l|k)$-selection-universal in $\mathcal{S}$. This is trivially true when $S \in \mathfrak{F}^{\mathcal{M}}_{tail} \cup \mathfrak{F}^{\mathcal{M}}_{\omega^p}$, since each of $\mathcal{S}_{tail}$ and $\mathcal{S}_{\omega^p}$ has the selection property. When $S \in \mathfrak{F}^{\mathcal{M}}_{\omega_1}$, $\mathcal{M}_{\restriction S} \cong (\omega_1, <, \epsilon_{l_1})$ is the unique member of $\mathcal{S}_{\omega_1}$ up to isomorphism, hence selection universal in it. Assume $\mathcal{M}_{\restriction S} \in \mathcal{S}_{>\omega^p}$. Let $\Phi \in \mathfrak{Form}_{n,l}$ and let $\rho \in \mathfrak{Form}_{k,l}$ select Φ in $\mathcal{M}'_S$. Then $\mathcal{M}'_S \models \text{uni-ax}(\rho, \Phi)$, which means $(\alpha'_S, <) \models \text{uni-ax}(\rho^*, \Phi^*)$. But, $\text{qd}(\text{uni-ax}(\rho^*, \Phi^*)) \leq \max\{n, k\} + 2l$. Since α'_S is a multiple of ω^q, it follows $(\omega^\omega, <) \models \text{uni-ax}(\rho^*, \Phi^*)$. By Proposition 3.20, Φ^* has a selector over Ω. By Lemma 4.4, this means Φ has a selector over $\Omega^\frown \epsilon_{l_1}$ which is $\mathcal{S}_{>\omega^p}$ up to isomorphism. We see then that $\mathfrak{S}'$ is a sequence as required in the Necessity-of-Sufficiency Lemma.

Finally, by definability below ω^ω, $\mathcal{P}^{<\delta}(\alpha)$ is definable in $(\alpha, <)$: $P \subseteq \alpha$ belongs to this class iff $\theta^{<\delta}_{\text{def}}$ holds in $(\alpha, <)$ restricted to P. Let $\text{uni-ax}^{<\delta}(\psi, \varphi)$ denote the conjunction of the sentences appearing in the definition of uniformization, but where the domain variables $\bar{X}$ are *restricted* to range over $\mathcal{P}^{<\delta}(\alpha)$. Then ψ is a δ-uniformizer for φ in $(\alpha, <)$ iff $(\alpha, <) \models \text{uni-ax}^{<\delta}(\psi, \varphi)$. Since the monadic theory of $(\alpha, <)$ is decidable, we can decide whether this is the case, i.e. whether φ has a δ-uniformizer in $(\alpha, <)$. Hurray! □

5. The selection problem over definable classes of countable ordinals

Given a sentence π and a formula $\varphi(\bar{Y})$, we would like to know whether φ has a selector over the class $\{(\alpha, <) \mid \alpha \in \omega_1 \setminus 1 \wedge (\alpha, <) \models \pi\}$ of countable ordinals satisfying π. In this section, we show that this can be decided and a selector – if one exists – can be computed. Thus, we may decide whether φ has a selector over the class of all countable ordinals, of countable limit ordinals, etc. In fact, we prove something slightly more general. We show that given a *formula* $\pi(x)$ and $\delta \leq \omega_1$, we can decide whether φ has a selector over $\{(\alpha, <) \mid \alpha \in \delta \setminus 1 \wedge (\delta, <) \models \pi(\alpha)\}$. This is indeed more general. For example, ω^ω is not a definable ordinal, but $\{\omega^\omega\}$ *is* definable in $(\omega^\omega + \zeta, <)$ for any $\zeta < \omega^\omega$.

We begin with a standard fact about n-types.

Lemma 5.1. *Let $n, r \in \omega$, $\mathcal{M}$ a structure. Then* $\text{type}^n(\mathcal{M})$ *determines* $\text{type}^n(\mathcal{M}^\frown \langle \underbrace{\varnothing, \ldots, \varnothing}_{r \text{ times}} \rangle)$.

Lemma 5.2. *Let $\mathcal{C}$ be a class of structures and $\varphi(\bar{X}, \bar{Y})$, $\psi_0(\bar{X}, \bar{Y}), \ldots, \psi_{m-1}(\bar{X}, \bar{Y})$ formulas. Suppose that for each $\mathcal{M} \in \mathcal{C}$, there exists $i < m$ such that ψ_i uniformizes φ in $\mathcal{M}$. Then φ has a uniformizer ψ over $\mathcal{C}$. In fact, ψ can be computed from $\varphi, \psi_0, \ldots, \psi_{m-1}$.*

Proof. Let $\psi := \bigwedge_{i<m} \left((\text{uni-ax}(\psi_i, \varphi) \wedge \bigwedge_{j<i} \neg\, \text{uni-ax}(\psi_j, \varphi)) \to \psi_i\right)$. □

Proposition 5.3. *There is an algorithm that solves the selection problem over the class $\{(\alpha, <) \mid \alpha \in \delta \setminus 1 \wedge (\delta, <) \models \pi(\alpha)\}$ for any formula $\pi(x)$ and $\delta \leq \omega_1$.*

Proof. Let $\pi(x)$ and δ be given. Denote by $\mathcal{C}$ the class appearing in the proposition. Fix a formula $\varphi(\bar{Y})$. Note that if $\delta < \omega^\omega$, then Proposition 1.3 solves our problem. We may assume then $\delta \geq \omega^\omega$. In fact, we begin by assuming $\delta \in \{\omega^\omega, \omega_1\}$.

Set $l := \lg(\bar{Y})$ and let

$$\varphi'(x, \bar{Y}) := \pi(x) \wedge \bigwedge_{i<l} (Y_i \subseteq [0, x)) \wedge \text{“}\varphi(\bar{Y}) \text{ holds restricted to } [0, x)\text{”}.$$

We claim φ has a selector over $\mathcal{C}$ iff φ' has a uniformizer in $(\delta, <)$.

If $\psi(\bar{Y})$ selects $\varphi(\bar{Y})$ over $\mathcal{C}$, let $\psi'(x, \bar{Y})$ be identical to φ', except that where φ appears in the latter, ψ appears in the former. Then ψ' uniformizes φ' in $(\delta, <)$. Conversely, assume some $\psi'(x, \bar{Y})$ uniformizes φ' in $(\delta, <)$. Let $\mathcal{C}' := \{\alpha \in \mathcal{C} \mid (\delta, <) \models \exists \bar{Y} \varphi'(\alpha, \bar{Y})\}$. Fix $\alpha \in \mathcal{C}'$. Since ψ' is a uniformizer, $\mathcal{D}(\psi'(\alpha, \bar{Y}), \delta)$ has a unique member. Denote it by $\bar{P}_\alpha$. Let $k := \text{qd}(\psi')$, $\tau'_\alpha := \text{type}^k(\alpha, <, \varnothing, \bar{P}_\alpha)$, and $\tau^* := \text{type}^k(\delta, <, \{0\}, \epsilon_l)$ where $\epsilon_l := \langle \underbrace{\varnothing, \ldots, \varnothing}_{l \text{ times}} \rangle$. Since $\delta \in \{\omega^\omega, \omega_1\}$,

$$(\delta, <, \{\alpha\}, \bar{P}_\alpha) \cong (\alpha, <, \varnothing, \bar{P}_\alpha) + (\delta, <, \{0\}, \epsilon_l).$$

Since also $(\delta, <, \{\alpha\}, \bar{P}_\alpha) \models \psi'$, we have $\tau'_\alpha + \tau^* \models \psi'$. Now, let $\psi := \bigvee_{\alpha \in \mathcal{C}'} \text{type}^k(\alpha, <, \bar{P}_\alpha)$. We claim ψ selects φ over $\mathcal{C}$.

First, since ψ' is a uniformizer for φ', $(\delta, <) \models \varphi'(\alpha, \bar{P}_\alpha)$ for each $\alpha \in \mathcal{C}'$. Hence, by definition of φ', $(\alpha, <) \models \varphi(\bar{P}_\alpha)$, i.e. $\text{type}^k(\alpha, <, \bar{P}_\alpha) \models \varphi$ (we assume $k \geq \text{qd}(\varphi)$). We see then that ψ is the disjunction of formulas which imply φ. Therefore, $\psi \models \varphi$. If $\alpha \in \mathcal{C}'$, then $\bar{P}_\alpha$ clearly satisfies ψ in $(\alpha, <)$. But, if $\alpha \in \mathcal{C}$, then any $\bar{P}$ satisfying φ in $(\alpha, <)$ is witness that in fact $\alpha \in \mathcal{C}'$. Thus, clause (3) of the definition of uniformization (1.1) holds. Finally, let some $\bar{P}$ satisfy ψ in $(\alpha, <)$. Then there is $\beta \in \mathcal{C}'$ such that $\text{type}^k(\alpha, <, \bar{P}) = \text{type}^k(\beta, <, \bar{P}_\beta)$. By Lemma 5.1, this means $\text{type}^k(\alpha, <, \varnothing, \bar{P}) = \text{type}^k(\beta, <, \varnothing, \bar{P}_\beta) = \tau'_\beta$. Thus,

$$\text{type}^k(\delta, <, \{\alpha\}, \bar{P}) = \text{type}^k(\alpha, <, \varnothing, \bar{P}) + \text{type}^k(\delta, <, \{0\}, \epsilon_l) = \tau'_\beta + \tau^* \models \psi'.$$

Since ψ' is a uniformizer, we must have $\bar{P} = \bar{P}_\alpha$.

We have thus reduced the selection problem over $\mathcal{C}$ to the special case of the bounded uniformization problem in $(\delta, <)$ where domain variables range over elements (equivalently, singleton subsets) of the domain. By Proposition 1.8, solvability of the former follows.

Now, to handle δ other than ω^ω and ω_1, note that the monadic theory of $(\delta, <)$ determines whether a formula $\psi(\bar{Y})$ selects φ over $\mathcal{C}$. Indeed, this is the case iff uni-ax(ψ, φ) holds restricted to $[0, \alpha)$ whenever $\alpha \in \mathcal{C}$. By Theorem 2.5, we may therefore assume $\delta = \omega^\omega + \zeta$ for some $\zeta \in \omega^\omega \setminus 1$. By the Segment Lemma (3.17), $\mathcal{C} \cap \omega^\omega$ is definable in $(\omega^\omega, <)$. Thus, by the case $\delta = \omega^\omega$, the selection problem over $\mathcal{C} \cap \omega^\omega$ is solvable. By Proposition 1.6, so is the selection problem over $\mathcal{C} \cap \{(\omega^\omega, <)\}$. To show the solvability of the selection problem over $\mathcal{C} \cap (\omega^\omega, \delta)$, proceed as follows. Set $n := \mathrm{qd}(\varphi)$. Let $\mathrm{Sel}^{\omega^\omega}_{n,l}$ be the set of (n, l)-Hintikka formulas that are both *satisfiable* and *selectable* in $(\omega^\omega, <)$. By Proposition 1.6, we can compute $\mathrm{Sel}^{\omega^\omega}_{n,l}$. Let $\mathbb{T}_{tail} := \{\tau' \in H_{n,l} \mid \exists \tau \in \mathrm{Sel}^{\omega^\omega}_{n,l} .\tau + \tau' \models \varphi\}$. By the Addition and Hintikka Lemmas, $\mathbb{T}_{tail}$ is computable. Let $\mathcal{C}' := \{\alpha \in \mathcal{C} \cap (\omega^\omega, \delta) \mid (\alpha, <) \models \exists \bar{Y}\varphi\}$. We claim φ has a selector over $\mathcal{C} \cap (\omega^\omega, \delta)$ iff

(Sel) for every $\alpha \in \mathcal{C}'$, $\bigvee \mathbb{T}_{tail}$ is satisfied in $(\alpha - \omega^\omega, <)$.

Furthermore, when this is the case, we show how to construct a selector for φ.

Suppose first ψ selects φ over $\mathcal{C} \cap (\omega^\omega, \delta)$. Fix $\alpha \in \mathcal{C}'$ and let $\bar{P}$ be the unique member of $\mathcal{D}(\psi, \alpha)$. If we let $\tau := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction[0,\omega^\omega)})$ and $\tau' := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction[\omega^\omega,\alpha)})$, then $\tau + \tau' \models \varphi$ and τ' is satisfied in $(\alpha - \omega^\omega, <)$. By the Segment Lemma, $\bar{P} \cap \omega^\omega$ is definable in $(\omega^\omega, <)$. But, $\bar{P} \cap \omega^\omega$ satisfies τ in $(\omega^\omega, <)$, so $\tau \in \mathrm{Sel}^{\omega^\omega}_{n,l}$. Assume conversely that (Sel) holds. Define a selector ψ for φ over $\mathcal{C} \cap (\omega^\omega, \delta)$ as follows. For each $\tau \in \mathrm{Sel}^{\omega^\omega}_{n,l}$, compute a Ψ_τ which selects it in $(\omega^\omega, <)$. For each $\tau \in \mathbb{T}_{tail}$, compute a Ψ'_τ which selects it over the class of all ordinals $< \zeta$ (by Proposition 1.3, this can be done). Note that ω^ω is definable in every $\alpha \in \mathcal{C} \cap (\omega^\omega, \delta)$ as the least $\beta < \alpha$ such that $\alpha - \beta \leq \zeta$. Fix an ordering $\prec$ on $\mathrm{Sel}^{\omega^\omega}_{n,l} \times \mathbb{T}_{tail}$. Let $\psi'(\bar{Y})$ say:

"For the $\prec$-minimal $(\tau, \tau') \in \mathrm{Sel}^{\omega^\omega}_{n,l} \times \mathbb{T}_{tail}$ such that $\exists \bar{Y} \tau'$ holds:
Ψ_τ holds restricted to $[0, \omega^\omega)$ and $\Psi'_{\tau'}$ holds restricted to $[\omega^\omega, \alpha)$."

Set $\psi := \exists \bar{Y}\varphi \wedge \psi'$. The reader will show that ψ is a selector for φ over $\mathcal{C}$.

Notice also that condition (Sel) is expressible by a sentence in the monadic theory of $(\delta, <)$ and hence, decidable. Thus, the selection problem over $\mathcal{C} \cap (\omega^\omega, \delta)$ is solvable.

By Lemma 5.2, solvability of the selection problem over $\mathcal{C} \cap \omega^\omega$, $\mathcal{C} \cap \{(\omega^\omega, <)\}$ and $\mathcal{C} \cap (\omega^\omega, \delta)$ implies solvability of this problem over $\mathcal{C}$. Our proof is therefore complete. □

6. The selection property for subclasses of ω^ω

We begin this section by presenting a simple combinatorial criterion for a class $\mathcal{C} \subseteq \omega^\omega$ to have the selection property. We then deduce the existence of finite expansions of $(\omega^\omega, <)$ having this property.

6.1. *Criterion for a subclass of ω^ω to have the selection property*

The next lemma and proposition are, respectively, Corollaries 6.1 and 3.9 of [9].

Lemma 6.1. *Let $n, l \in \omega$ and $\phi \in \mathfrak{Form}_{n,l}$ which is satisfied in $(\omega^\omega, <)$. Then we can compute $\tau_{sel}, \tau_{suf} \in H_{n,l}$ such that:*

(a) *τ_{sel} is satisfiable and selectable in $(\omega^\omega, <)$,*
(b) *τ_{suf} is satisfiable in $(\omega^\omega, <)$, and*
(c) *$\tau_{sel} + \tau_{suf} \models \phi$.*

Proposition 6.2. *The formula saying "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$.*

Recall that for ordinals $\mu > 0$ and α, the μ-*part* of α is the maximal multiple of μ smaller than or equal to α (Definition 4.5).

Proposition 6.3. *A class $\mathcal{C} \subseteq \omega^\omega$ has the selection property iff*

(Sel) $\forall p \in \omega \exists N(p) \in \omega \forall \alpha \in \mathcal{C} \big($*the ω^p-part of α is not a multiple of $\omega^{N(p)}$*$\big)$.

If in addition, $N(p)$ is computable from p, then selectors are computable over $\mathcal{C}$.

Proof. For each ordinal α and $k \in \omega$, we denote by α_k the ω^k-part of α.

Suppose first condition (Sel) holds. Fix $n, l \in \omega$ and $\varphi \in \mathfrak{Form}_{n,l}$. Let $p := \mathfrak{p}(n + l) + 1$. Pick $N := N(p)$ as in (Sel). Then $N \geq p$.[14] By Proposition 1.3, $\mathcal{C} \cap \omega^N := \{\alpha \in \mathcal{C} \mid \alpha < \omega^N\}$ has the selection property and selectors are computable over it. Therefore, by Lemma 5.2, it suffices that we show $\mathcal{C} \setminus \omega^N := \{\alpha \in \mathcal{C} \mid \alpha \geq \omega^N\}$ has the selection property. Hence, we may assume that for every $\alpha \in \mathcal{C}$, $\alpha \geq \omega^N$. We may further assume that φ is satisfied in every $\alpha \in \mathcal{C}$. We claim that for every $\alpha \in \mathcal{C}$, there are $\tau, \tau' \in H_{n,l}$ such that:

14 Except perhaps when $\mathcal{C} \subseteq \omega^p$. But Proposition 1.3 takes care of this trivial case.

1. τ is satisfiable and selectable in $(\omega^\omega, <)$,
2. τ' is satisfiable in $(\alpha - \alpha_N, <)$,
3. $\tau + \tau' \models \varphi$.

Indeed, fix $\alpha \in \mathcal{C}$. Pick some $\bar{P} \in \mathcal{D}(\varphi, \alpha)$ and let $\tau_p := \text{type}^n((\alpha, <, \bar{P})_{\restriction[0,\alpha_p)})$. Since α_p is a multiple of ω^p, it is $(n+l)$-equivalent to ω^ω, so τ_p is satisfied in ω^ω. Pick τ_{sel}, τ_{suf} as in Lemma 6.1, setting $\phi := \tau_p$. Let $\tau := \tau_{sel}$. Next, write $\tau_{\text{tail}} := \text{type}^n((\alpha, <, \bar{P}))_{\restriction[\alpha_p,\alpha)}$ and let $\tau' := \tau_{suf} + \tau_{\text{tail}}$ (if $\alpha = \alpha_p$, drop τ_{tail}). Note that $\alpha_p - \alpha_N$ is also a non-0 multiple of ω^p (by choice of N, we cannot have $\alpha_N = \alpha_p$). Therefore, τ_{suf} is satisfied in $\alpha_p - \alpha_N$, so that τ' is indeed satisfied in $\alpha - \alpha_N$. Also, $\tau + \tau' = \tau_{sel} + \tau_{suf} + \tau_{\text{tail}} = \tau_p + \tau_{\text{tail}} \models \varphi$, as desired.

Let $\tau \in H_{n,l}$. If τ is selectable in $(\omega^\omega, <)$, Proposition 3.20 lets us compute a selector Ψ_τ for τ over the class of all countable multiples of ω^p. In particular, Ψ_τ selects τ in α_N for every $\alpha \in \mathcal{C}$. Using Proposition 1.3, we can also compute a selector Ψ'_τ for τ over the class of ordinals below ω^N. Fix an ordering $\prec$ of $H_{n,l} \times H_{n,l}$ (a finite set). A selector for φ over $\mathcal{C}$ is the $\psi(\bar{Y})$ which says:

"For the $\prec$-least pair $\tau, \tau' \in H_{n,l}$ which satisfies (1)–(3),
Ψ_τ holds restricted to $[0, \alpha_N)$, while Ψ'_τ holds restricted to $[\alpha_N, \alpha)$."

A quick review of the proof would convince the reader of the computability of ψ from φ *and* N, which justifies the "If in addition …" clause.

Now, assume (Sel) fails. Then we can pick $p \in \omega$ such that $\{\alpha_p \mid \alpha \in \mathcal{C}\}$ is fat. Let $\varphi(Y)$ say (in every $\alpha \in \mathcal{C}$): "Y is an unbounded ω-sequence in α_p". Fix any formula $\psi(Y)$. Then there is $\alpha \in \mathcal{C}$ which is a multiple of ω^N where $N := \mathfrak{p}(\text{qd}(\text{uni-ax}(\psi, \varphi)))$. If ψ selected φ in α, then by the $\mathfrak{p}$-Lemma (3.19), it would also select φ in $(\omega^\omega, <)$. But, by Proposition 6.2, this cannot be the case. Thus, ψ is *not* a selector for φ over $\mathcal{C}$. □

Example. Both of the following classes have the selection property and selectors are computable over them (in both let $N(p) := p + 1$ for every $p \in \omega$):

(1) The unbounded ω-sequence $\{1, \omega + 1, \omega^2 + \omega + 1, \omega^3 + \omega^2 + \omega + 1, \ldots\}$.

(2) The $\mathcal{C} \subseteq \omega^\omega$ defined as follows. Let $\alpha < \omega^\omega$ and write $\alpha = \sum_{i \le n} \omega^{n-i} a_{n-i}$ with $n, a_i \in \omega$ and $a_n \neq 0$.[15] Then $\alpha \in \mathcal{C}$ iff for every $i \le n$, $a_i \neq 0$. Note that $\mathcal{C}$ has order-type ω^ω.

6.2. Finite expansion of $(\omega^\omega, <)$ having the selection property

From the last proposition, we now deduce Proposition 1.10. To do so, note first that Proposition 2.14 allows us also to define *multiplication* of an Hintikka formula by any linear order.

Definition 6.4. Let $n, l \in \omega$, $\tau_1 \in H_{n,l}$, and $\mathcal{I}$ a linear order. The $\mathcal{I}$-*multiple* of τ_1, denoted $\tau_1 \otimes \mathcal{I}$, is an element of $H_{n,l}$ such that whenever $\mathcal{M}$ is an l-structure with $\text{type}^n(\mathcal{M}) = \tau_1$,

$$\text{type}^n(\mathcal{M} \otimes \mathcal{I}) = \tau_1 \otimes \mathcal{I}.$$

And the Composition Theorem yields the following lemma from [10]:

Lemma 6.5. *For every linear order $\mathcal{I}$, $\lambda n, l \in \omega.\lambda\tau_1 \in H_{n,l}.\tau_1 \otimes \mathcal{I}$ is recursive in* $\text{MTh}(\mathcal{I})$.

The following clearly implies Proposition 1.10.

Corollary 6.6. *Let $P^\star := \{1, \omega + 1, \omega^2 + \omega + 1, \omega^3 + \omega^2 + \omega + 1, \ldots\}$. Then:*

(a) *$(\omega^\omega, <, P^\star)$ has the selection property;*
(b) *for any formula $\varphi(X, \bar{Y})$, a selector for φ in $(\omega^\omega, <, P^\star)$ is computable; and*
(c) *the monadic theory of $(\omega^\omega, <, P^\star)$ is decidable.*

Proof. We prove (a) in somewhat greater generality. Pick any unbounded $P \subseteq \omega^\omega$ such that $\mathcal{C}_P := \{[\alpha, \alpha') \mid \alpha < \alpha'$ are successive elements of $P \cup \{0\}\}$ has the selection property and $\text{otp}(P) < \omega^\omega$ (both assumptions hold for $P^\star$; the first by Proposition 6.3). We claim that $\mathcal{M}_P := (\omega^\omega, <, P)$ has the selection property. Let θ be a splitting of $\mathcal{M}_P$ whose segments are the members of $\mathcal{C}_P$. Then $\mathcal{I}_{\mathcal{M}_P/\theta}$ is isomorphic to an ordinal $< \omega^\omega$. In particular, it has the uniformization property. Also, members of $\text{Smd}_{\{\mathcal{M}_P\}/\theta}$ have the form $(\beta, <, \{0\})$ where $\beta \in \mathcal{C}_P$. Since $\mathcal{C}_P$ has the selection property and $\{0\}$ is definable in $(\beta, <)$, $\text{Smd}_{\{\mathcal{M}_P\}/\theta}$ has the selection property. By Corollary 3.13 (to the Inheritance Lemma), so does $\mathcal{M}_P$.

(b) Proposition 1.3 tells us that in (a) selectors are computable over $\text{Ind}_{\{\mathcal{M}_P\}/\theta}{}^\frown\text{TyPart}$. On the other hand, when $P = P^\star$, we can take $N(p) := p + 1$ (for every $p \in \omega$) in Proposition 6.3. In particular, $N(p)$ is computable from p, which means that selectors are computable over $\text{Smd}_{\{\mathcal{M}_{P^\star}\}/\theta}$. By the Inheritance Lemma again, this also holds in $\mathcal{M}_{P^\star}$.

(c) Fix $n \in \omega$. We show how to compute $\text{type}^n(\mathcal{M}_{P^\star})$. Use the $\mathfrak{p}$-Lemma to compute $p \in \omega$ such that any two non-0 countable multiples of ω^p are $(n+1)$-equivalent. Let α_0 be the least member of $P^\star$ greater than ω^p. Then $\text{type}^n(\mathcal{M}_{P^\star}) =$

[15] Recall that this presentation is unique.

$\mathrm{type}^n(\mathcal{M}_{P^\star\restriction[0,\alpha_0)}) + \mathrm{type}^n(\mathcal{M}_{P^\star\restriction[\alpha_0,\omega^\omega)})$. The left summand here is computable because $P^\star \cap \alpha_0$ is finite and hence, by Lemma 4.6, definable in $(\alpha_0, <)$, and because $\mathrm{MTh}(\alpha_0, <)$ is decidable. To handle $\mathrm{type}^n(\mathcal{M}_{P^\star\restriction[\alpha_0,\omega^\omega)})$, fix $\alpha \in P^\star \setminus \alpha_0$ and let α' be its successor in $P^\star$. Then, by definition of $P^\star$,

$$\mathcal{M}_{P^\star\restriction[\alpha,\alpha')} = \mathcal{M}_{P^\star\restriction[\alpha,(\alpha')_p)} + \mathcal{M}_{P^\star\restriction[(\alpha')_p,\alpha')} \cong ((\alpha')_p - \alpha, <, \{0\}) + (\zeta, <, \emptyset),$$

where $(\alpha')_p$ is the ω^p-part of α' and $\zeta := \omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1$. Since $(\alpha')_p - \alpha$ is a multiple of ω^p, $((\alpha')_p, <) \equiv^{n+1} (\omega^p, <)$. From this it easily follows that $((\alpha')_p - \alpha, <, \{0\}) \equiv^n (\omega^p, <, \{0\})$. Thus, $\mathrm{type}^n(\mathcal{M}_{P^\star\restriction[\alpha_0,\omega^\omega)}) = \left(\mathrm{type}^n(\omega^p, <, \{0\}) + \mathrm{type}^n(\zeta, <, \emptyset)\right) \otimes \omega$. By Theorem 2.6, both n-types inside the brackets here are computable, hence so is their sum. Since $\mathrm{MTh}(\omega, <)$ is decidable, Lemma 6.5 computes for us $\mathrm{type}^n(\mathcal{M}_{P^\star\restriction[\alpha_0,\omega^\omega)})$, proving (c). □

Appendix. Uniformization below ω^ω

Here we apply the Inheritance Lemma to prove Proposition 1.3.

The following proposition was stated in [5] and attributed to [1]. However, [1] deals with the Church synthesis problem, and not with the uniformization problem (for a clarification of the difference between the two, see e.g. [9]). A detailed proof was eventually supplied in [7]. It uses Ramsey theory (and the composition method) to reduce uniformization in $(\omega, <)$ to uniformization over finite ordinals.

Proposition A.1. *$(\omega, <)$ has the uniformization property and the uniformization problem in $(\omega, <)$ is computable.*

Next, we need a lemma from the 'folklore'.

Lemma A.2. *$\{(n, <) \mid n \in \omega \setminus 1\}$ has the uniformization property and the uniformization problem over this class is computable.*

Proof. Use the fact that the lexicographical ordering of tuples (of a given length l_2) of subsets of n is definable and a well-order. □

The following is Proposition 1.3.

Proposition A.3. *For $k \in \omega \setminus 1$, $\{(\alpha, <) \mid \alpha \in (0, \omega^k]\}$ has the uniformization property and uniformization is computable over this class.*

Proof. We proceed by induction on $k \in \omega \setminus 1$. Let $k = 1$. Since both the class of finite ordinals and $\{(\omega, <)\}$ have the uniformization property and uniformizers are computable over both (by A.1 and A.2), Lemma 5.2 yields the same for $\{(\alpha, <) \mid \alpha \in (0, \omega]\}$.

Let $k \in \omega \setminus 1$ and assume our claim holds for this k. To prove the lemma, we fix an $l_1 \in \omega$ and show that $\mathcal{C} := \mathrm{Exp}_{l_1}^{\leq\omega^{k+1}}(\{(\alpha, <) \mid \alpha \in (0, \omega^{k+1}]\})$ has the selection property and selection is computable over this class.

Write a formula $\theta(x, y)$ which says: "There are no multiples of ω^k strictly between x and y".[16] Then θ is a splitting of $\mathcal{C}$ and for $\mathcal{M} \in \mathcal{C}$ with domain α,

$$I_{\mathcal{M}/\theta} = \{[\omega^k \cdot i, \min\{\omega^k \cdot (i+1), \alpha\}) \mid i \in \omega \wedge \omega^k \cdot i < \alpha\}.$$

In particular, $\mathcal{I}_{\mathcal{M}/\theta}$ is isomorphic to an ordinal $\leq \omega$ (in fact, to a finite ordinal unless $\alpha = \omega^{k+1}$). By the case $k = 1$, $\mathrm{Ind}_{\mathcal{C}/\theta}{}^\frown\mathrm{TyPart}$ has the selection property and selectors are computable over it. Also, the domain of every segment has order-type $\leq \omega^k$ (in fact, all have order-type ω^k save perhaps for the 'top' segment, if one exists). In other words,

$$\mathrm{Smd}_{\mathcal{C}/\theta} \subseteq \mathrm{Exp}_{l_1}^{\leq\omega^k}(\{(\beta, <) \mid \beta \in (0, \omega^k]\}).$$

By the inductive assumption, the latter has the selection property and selection is computable over it. The Inheritance Lemma now yields the same for $\mathcal{C}$, to our heart's delight. □

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[16] Equivalently: "x and y have the same ω^k-part".