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The theory of finite structures is very different from the theory of arbitrary structures

Hilbert Calculus

Axioms[a]

Ax1 $A \to (B \to A)$

Ax2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Ax3 $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Ax4 $(\forall x A(x)) \to A\{t/x\}$, where t is a term.

Ax5 $(\forall x (A \to B)) \to (A \to \forall x B)$, where x is not free in A.

Inference Rules

MP Derive B from A and $A \to B$.

Gen Derive $\forall x A$ from A.

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[a] We do not distinguish between formulas with the same skeleton

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Predicate Calculus

Theorem Every consistent set of formulas is a subset of a Complete Henkin consistent set of formulas.

Theorem Every Complete Henkin consistent set of formulas holds (in a Herbrand Structure).

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Show that $\cup_n \Gamma_n$ is a consistent and Σ-complete.

Henkin Sets of Formulas

Definition Γ has Henkin property for Σ if for every sentence $\psi \in \Gamma$ of the form $\neg\forall x\varphi$ in the signature Σ there is a constant c such that $\neg\varphi\{c/x\} \in \Gamma$ (sign. of $\Gamma \supseteq \Sigma$).

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Apply Lemma to all Σ-sentences of the form $\neg\forall x\varphi$ in Γ

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By structural induction on sentences show that

$$\varphi \in \Gamma \text{ iff } [||\varphi||]^M = true$$