On the Decidability of Continuous Time Specification Formalisms

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Abstract

We consider an interpretation of monadic second-order logic of order in the continuous time structure of finitely variable signals and show the decidability of monadic logic in this structure. The expressive power of monadic logic is illustrated by providing a straightforward meaning preserving translation into monadic logic of three typical continuous time specification formalism: temporal logic of reals, restricted duration calculus and the propositional fragment of mean value calculus. As a by-product of the decidability of monadic logic we obtain that the above formalisms are decidable even when extended by quantifiers.

Keywords: Monadic second-order logic, temporal logic, duration calculus, decidability.

1 Introduction

In recent years systems whose behaviour changes in continuous (real) time have been extensively investigated. Hybrid and control systems are prominent examples of real-time systems.

A number of formalisms for specification of real-time behaviour were suggested in the literature. Some of these formalisms (e.g. timed automata [1]) extend discrete time formalisms by introducing metrical real-time constraints, others (e.g. temporal logic of reals [2]) are defined by providing continuous (or dense) time interpretation for the modalities studied in the discrete cases, yet others (e.g. duration calculus [5]) are based on ideas that were not popular among the formalisms for specification of discrete time behaviour.

It is worthwhile to distinguish two aspects of real-time specifications: (a) Metric aspects which deal with the distance between moments of real time; (b) Properties of the order on the real numbers, e.g. the order is dense and Dedekind closed. In this paper metric aspects of specification are not considered. Specifications that use only the order relation on the reals will be called continuous time specifications and are investigated in the sequel. In the conclusing section we comment about metrical extensions.

A run of a real-time system is represented by a function from non-negative reals into a set of values—the instantaneous states of a system. Such a function will be called a signal. Usually, there is a further restriction on behaviour of continuous time systems. For example a function that gives a value q_0 for the rationals and value q_1 for the irrationals is not accepted as a 'legal' signal.

A requirement that is often imposed in the literature is that in every bounded time interval a system can change its state only finitely many times. This requirement is called finite variability (or non-Zeno) requirement. It is clear that finite variability requirement is not a metric requirement.

Recall that the language $L_2^<$ of monadic second-order logic of order contains individual variables, second-order variables and the binary predicate $<$. In the discrete time structure ω (this structure will be defined precisely in Section 2.3), the individual variables are interpreted as natural numbers, the second-order variables as monadic predicates (monadic functions from the natural numbers into the Booleans), and $<$ is the standard order on the set of natural numbers.

In this paper we consider an interpretation of monadic logic in the continuous time structure of finitely variable signals. In this structure the individual variables range over real numbers, the second order variables range over finitely variable Boolean signals, and $<$ is the standard order relation on the set of real numbers. Note that metric properties of reals cannot be specified in this logic.

First, we show that the $L_2^<$ theory of the finitely variable signal structure is decidable. The result is significant due to the fact that many specification formalisms for reasoning about real time which were considered in the literature can be effectively embedded in $L_2^<$. In order to illustrate the expressive power of $L_2^<$ over finitely variable signals, we consider the following three formalisms for specifying non-metric properties of a continuous time behaviour:

- restricted duration calculus—RDC [4].
- propositional mean value calculus—PMVC [6, 20, 11].
- temporal logic of reals—TLR [2].

We recall the definition of these formalisms and provide meaning preserving compositional translations from the above formalisms into the first-order fragment of $L_2^<$. These translations are directly obtained by a formalization of the semantical definitions for RDC, PMVC and TLR.

As a by-product of the decidability of $L_2^<$, we obtain a simple and uniform proof of the decidability of the above formalisms. Decidability of TLR, RDC and PMVC are not novel. A tableau decision algorithm for TLR was described in [8]; the presentation of this algorithm and the proof of its correctness is quite long. The decidability of RDC was proved in [4] and the decidability of PMVC was proved in [20]. The decision procedure for RDC and PMVC appeals to automata theoretical methods. Neither the decision algorithm for TLR, nor the decision algorithms for RDC and PMVC can be generalized to treat quantifiers. However, from the decidability of second-order monadic logic we easily obtain that the extensions of RDC, PMVC and TLR by the quantifiers are also decidable.

The rest of this paper is organized as follows. In Section 2 the syntax and semantics of monadic second-order logic of order is provided and classical theorems about important structures for this logic are stated. In Section 3 we show that the set of monadic second-order logic sentences that are true in the finitely variable signal structure is decidable. In Section 4 we provide compositional translations of the restricted duration calculus, the propositional fragment of mean value calculus and the temporal logic of reals into monadic logic. In Section 5 we comment on some related results.

2 Monadic Second-order theory of order

In this section we recall the definitions of the syntax and the semantics of monadic second-order theory of order.

2.1 Syntax

The language $L_2^<$ of monadic second-order theory of order has a set Var_1 of individual variables, a set Var_2 of second-order variables, a binary predicate $<$, the usual propositional connectives and first and second-order quantifiers.

We will use t, u, v for individual variables and x, y for second-order variables.

The atomic formulas of $L_2^<$ are formulas of the form: $t < u$ and $x(t)$. The formulas are constructed from atomic formulas by logical connectives and first and second-order quantifiers.

We will write $F(x, y, t, u)$ to indicate that the free variables of a formula F are among x, y, t, u.

2.2 Semantics

A structure $K = \langle A, B, <_K \rangle$ for $L_2^<$ consists of a set A partially ordered by $<_K$ and a set B of monadic functions from A into $BOOL$.

An environment α for individual variables is a function from the set of individual variables into A and an environment η for the second-order variables is a function from the set of second-order variables into B. Below the satisfiability relation $(\alpha, \eta) \models \psi$ is defined by induction on the structure of $L_2^<$ formulas.

DEFINITION 2.1
(Semantics of $L_2^<$ formulas)

1. $(\alpha, \eta) \models t < u$ if $\alpha(t) <_K \alpha(u)$.
2. $(\alpha, \eta) \models x(t)$ if $\eta(x)$ maps $\alpha(t)$ to *TRUE*.
3. $(\alpha, \eta) \models \psi_1 \wedge \psi_2$ if $(\alpha, \eta) \models \psi_1$ and $(\alpha, \eta) \models \psi_2$.
4. $(\alpha, \eta) \models \neg\psi$ if not $(\alpha, \eta) \models \psi$.
5. $(\alpha, \eta) \models \exists^1 t.\psi$ if there exists α' such that $\alpha(u) = \alpha'(u)$ for all $u \neq t$ and $(\alpha', \eta) \models \psi$.
6. $(\alpha, \eta) \models \exists^2 x.\psi$ if there exists η' such that $\eta(y) = \eta'(y)$ for all $y \neq x$ and $(\alpha, \eta') \models \psi$.

Notation: (a) In (5) the first-order existential quantifier $\exists^1$ was defined and in (6) the second-order existential quantifier $\exists^2$ was defined. The symbol $\exists$ will be used for both these quantifiers in the sequel; the ambiguity will always be resolved by a context. If $\exists$ is followed by an individual (second-order) variable it will refer to the first (second)-order existential quantifier. (b) We should have used $\models_K^{L_2^<}$ for the satisfiability relation in a structure K of language $L_2^<$, however, in the sequel the ambiguity will always be resolved by a context.

2.3 Examples of structures for $L_2^<$

In this section we present three classical structures for $L_2^<$ and recall some remarkable theorems.

We will use the following:

Notation: $\mathbf{N}$ will be used for the set of natural numbers and $<_N$ for the standard order on $\mathbf{N}$. $\mathbf{R}$ (respectively $\mathbf{R}^{\geq 0}$) will be used for the set of real (respectively, non-negative real) numbers and $<_R$ for the standard order on $\mathbf{R}$.

2.3.1 Structure ω

The structure $\omega = \langle \mathbf{N}, 2^{\mathbf{N}}, <_N \rangle$, where $2^{\mathbf{N}}$ is the set of all monadic functions from $\mathbf{N}$ into $BOOL$.

THEOREM 2.2
(Buchi [3]) The set of $L_2^<$ sentences true in ω is decidable.

2.3.2 The structure of reals

Shelah considered the structure $M = \langle \mathbf{R}, 2^{\mathbf{R}}, <_R \rangle$, where $2^{\mathbf{R}}$ is the set of all monadic functions from $\mathbf{R}$ into $BOOL$.

THEOREM 2.3
(Shelah [18]) The set of $L_2^<$ sentences true in M is undecidable.

2.3.3 F Structure for $L_2^<$

Rabin considered the structure $F = \langle \mathbf{R}, F_\sigma, <_R \rangle$, where F_σ is the set of monadic functions from $\mathbf{R}$ into $BOOL$ such that $x \in F_\sigma$ iff either $\{\tau : x(\tau) = TRUE\}$ or $\{\tau : x(\tau) = FALSE\}$ is a countable union of closed sets.

THEOREM 2.4
(Rabin [12]) The set of $L_2^<$ sentences true in F is decidable.

3 Finitely variable signal structure

Below we define finitely variable signal structure.

DEFINITION 3.1
A function h from the non-negative reals into the set $BOOL$ is called a Boolean finitely variable signal if there exists an unbounded increasing sequence $\tau_0 = 0 < \tau_1 < \tau_2 \ldots < \tau_n < \ldots$ such that h is constant on every interval (τ_i, τ_{i+1}). For a finite set Σ, the notion of Σ-signal is defined similarly.

The word 'signal' will often stand below for 'finitely variable signal'.

We say that a signal x is right continuous at t iff there is $t_1 > t$ such that $x(t) = x(t')$ for all t' which satisfies $t < t' < t_1$.

We say that a signal x is left continuous at t iff $t = 0$ or there is $t_1 < t$ such that $x(t) = x(t')$ for all t' which satisfies $t_1 < t' < t$.

We say that a signal is left (right) continuous iff it is left (right) continuous at t for every t.

Let *SIGNAL* (respectively RSIGNAL, or LSIGNAL) be the set of all Boolean finitely variable signals (respectively right continuous signal or left continuous signals). The signal structure Sig is defined as $Sig = \langle \mathbf{R}^{\geq 0},\ SIGNAL, <_R\ \rangle$, where $\mathbf{R}^{\geq 0}$ is the set of non-negative reals. The structures of right continuous signals and left continuous signals are defined similarly.

In [14] it was proved that monadic second-order logic over the structure of right continuous signals is decidable. Slightly modifying the proof for right continuous structures we obtain the following theorem.

THEOREM 3.2
The set of $L_2^<$ sentences true in the signal structure Sig is decidable.

PROOF. First, let us note that for the restriction of the structure F to non-negative reals, Theorem 2.4 still holds.

It is clear that if x is a finitely variable signal then $\{\tau \in \mathbf{R}^{\geq 0} \mid x(\tau) = TRUE\}$ and $\{\tau \in \mathbf{R}^{\geq 0} \mid x(\tau) = FALSE\}$ can be represented as a countable union of closed sets. Hence, every signal belongs to F_σ. It is also clear that $x \in F_\sigma$ is a signal if and only if it satisfies the formula $signal(x)$ defined as:

$$\begin{aligned} signal(x) \triangleq\ & \forall t.\exists t_1.t < t_1 \wedge \forall t_2.t < t_2 \leq t_1 \rightarrow (x(t_1) \leftrightarrow x(t_2)) \wedge \\ & \forall t.t > 0 \rightarrow \exists t_1.t_1 < t \wedge \forall t_2.t_1 < t_2 < t \rightarrow (x(t_1) \leftrightarrow x(t_2)). \end{aligned}$$

Below we provide an interpretation of the signal structure Sig inside structure F. (See [13] for the detailed description of the methods of interpretation.)

If A is a monadic second-order formula then the formula A^{Sig} obtained from A by relativizing all second-order quantifiers of A to signals is defined inductively on the structure of A by the following rules: (1) If A is without second-order quantifiers then $A^{Sig} = A$. (2) If $A = B \wedge C$ or $A = \neg B$ or $A = \exists^1 t.B$ then $A^{Sig} = B^{Sig} \wedge C^{Sig}$ or $A^{Sig} = \neg B^{Sig}$ or $A^{Sig} = \exists^1 t.B^{Sig}$, respectively. (3) If $A = \exists^2 x.B$ or $A = \forall^2 x.B$ then $A^{Sig} = (\forall^2 x.signal(x) \wedge B^{Sig})$ or $A^{Sig} = (\forall^2 x.signal(x) \rightarrow B^{Sig})$, respectively.

This relativization allows us to reduce the satisfiability of the formula A in the structure Sig to the satisfiability of the formula A^{Sig} in the structure F. In particular, if A is a closed formula, then $\models_{Sig}\ A$ if and only if $\models_F\ A^{Sig}$. Therefore, Theorem 3.2 follows from Theorem 2.4. ∎

One can is easily adapt this proof in order to show that the set of $L_2^<$ sentences true in the structure of left continuous signals is decidable.

4 Compositional translations into $L_2^<$

In this section we consider three formalisms: temporal logic of reals [2], a non-metrical subset of the duration calculus which is called RDC [4] and the propositional fragment of mean value calculus [6, 20, 11] (PMVC). We recall the definition of syntax and the semantics of TLR, RDC and PMVC and provide meaning preserving compositional translations of these formalisms into first-order fragment of monadic logic of order. As a by product we obtain that the above formalisms are decidable even when extended by the second-order quantifiers.

Throughout this section we will use the following

Notation: $D\{u'/u\}$ is a formula obtained from a formula D by the substitution of u' for all free occurrences of u in D.

4.1 Temporal logic of reals

The temporal logic of reals was proposed in [2]. It is based on the same set of modalities as linear temporal logic, however, these modalities are interpreted over the time domain of non-negative real numbers.

The set of TLR formulas is defined by the following grammar:

$D ::= x \mid \neg D \mid D \wedge D \mid \bigcirc D \mid \ominus D \mid D \,\mathcal{U}\, D$, where x ranges over the variables.

An environment maps variables into finitely variable signals. For an environment ρ and a real number r, the satisfiability relation $(\rho,\ r) \models_{TLR} D$ is defined as follows:

- $(\rho,\ r) \models_{TLR} x$ iff $\rho(x)(r)$ is true.
- $(\rho,\ r) \models_{TLR} \bigcirc D$ iff there exists $r_1 > r$ such that $(\rho,\ r_2) \models_{TLR} D$ for all $r_2,\ r < r_2 < r_1$.
- $(\rho,\ r) \models_{TLR} \ominus D$ iff there exists $r_1 < r$ such that $(\rho,\ r_2) \models_{TLR} D$ for all $r_2,\ r_1 < r_2 < r$.
- $(\rho,\ r) \models_{TLR} D_1 \,\mathcal{U}\, D_2$ iff there exists $r_1 > r$ such that $(\rho,\ r_1) \models_{TLR} D_2$ and $(\rho,\ r_2) \models_{TLR} D_1$ for all $r_2, r \leq r_2 < r_1$.
- $(\rho,\ r) \models_{TLR} D_1 \wedge D_2$ iff $(\rho,\ r) \models_{TLR} D_1$ and $(\rho,\ r) \models_{TLR} D_2$.
- $(\rho,\ r) \models_{TLR} \neg D$ iff not $(\rho,\ r) \models_{TLR} D$.

Let us fix an individual variable t. Let $D(x_1, \ldots x_n)$ be a TLR formula with free variables in the set $\{x_1 \ldots x_n\}$. Our translation will map D to $L_2^<$ formula $D'(x_1, \ldots x_n, t)$ with free second-order variables $x_1, \ldots x_n$ and free individual variable t.

The following theorem will hold for the translation Tr presented below.

THEOREM 4.1
Let D be a TLR formula. Then $(\rho,\ r) \models_{TLR} D$ iff $(\rho,\ [t \to r]\ ) \models Tr(D)$, where $[t \to r]$ is the environment that assigns to the first-order variable t the real number r.

The translation is defined as follows:

- $Tr(x)$ is defined as $x(t)$.
- $Tr(D_1 \wedge D_2)$ is defined as $Tr(D_1) \wedge Tr(D_2)$.
- $Tr(\neg D)$ is defined as $\neg Tr(D)$.
- $Tr(\bigcirc D)$ is defined as $\exists t_1 > t. \forall t_2.\ (t < t_2 < t_1) \to (D\{t_2/t\})$.
- $Tr(\ominus D)$ is defined as $\exists t_1 < t. \forall t_2.\ (t_1 < t_2 < t) \to (D\{t_2/t\})$.
- $Tr(D_1 \,\mathcal{U}\, D_2)$ is defined as $\exists t_1 > t. (D_2\{t_1/t\}) \wedge \forall t_2. (t \leq t_2 < t_1) \to (D_1\{t_2/t\})$.

It is easy to see that $Tr(D)$ is a first-order monadic formula and that Theorem 4.1 holds.

Let QTLR be the extension of TLR by the quantifiers over the variables, i.e. QTLR is obtained by adding to the syntax of TLR the clause: if D is a formula then $\exists x.D$ is a formula; the semantical clause for $\exists$-quantifier is defined precisely as for the standard second-order existential quantifier. The translation is extended to QTLR by the rule $Tr(\exists x.D)$ is $\exists x.Tr(D)$. It is immediate that Theorem 4.1 holds for QTLR.

As a consequence of Theorem 4.1 and Theorem 3.2 we obtain the following corollary.

COROLLARY 4.2
The extension of TLR by the quantifiers is decidable.

4.2 Restricted duration calculus

The restricted duration calculus is a fragment of duration calculus in which metric properties are ignored and state variables are interpreted as finitely variable Boolean signals.

The *state expressions* are constructed from state variables by Boolean operations. We will use S to range over the state expressions and X to range over the state variables.

The formulas of RDC are defined by the following grammar:

$$D ::= \lceil S \rceil \mid D \frown D \mid \neg D \mid D \wedge D \text{ , where } S \text{ ranges over state expressions.}$$

The atomic formulas of RDC are the formulas of the form $\lceil S \rceil$ and other formulas are obtained from atomic formulas by applying propositional connectives and the 'chop' operator $\frown$ of interval temporal logic [9, 7].

An environment maps the state variables to finitely variable signals. The meaning of a state expression S in an environment ρ is a signal $[|S|]\rho$ defined as follows:

- $[|x|]\rho$ is $\rho(x)$.
- $[|S_1 \wedge S_2|]\rho$ is $[|S_1|]\rho \& [|S_2|]\rho$, where & is the pointwise extension of the conjunction to signals, i.e. for every time moment τ the value of $[|S_1 \wedge S_2|]\rho$ at τ is the conjunction of the values of $[|S_1|]\rho$ and $[|S_2|]\rho$ at τ.
- $[|\neg S|]\rho$ is $\mathbf{not}[|S|]\rho$, where **not** is the pointwise extension of the negation to signals.

Given an environment ρ and real numbers b and e, the satisfiability of RDC formulas in the interval $[b, e]$. under environment ρ is defined as follows:

- $(\rho, [b, e]) \models_{RDC} \lceil S \rceil$ iff $b < e$ and $[|S|]\rho$ has the value $TRUE$ almost everywhere in the interval $[b, e]$, i.e., $\int_b^e [|S|]\rho = e - b$. However, since $[|S|]\rho$ is a finitely variable signal, this requirement is equivalent to the requirement that $[|S|]\rho$ receives the value $FALSE$ at a finite number of points in the interval $[b, e]$,
- $(\rho, [b, e]) \models_{RDC} D_1 \frown D_2$ iff $(\rho, [b, m]) \models_{RDC} D_1$ and $(\rho, [m, e]) \models_{RDC} D_2$ for some m in the interval $[b, e]$. Thus, $D_1 \frown D_2$ is true in an interval if the interval can be partitioned ('chopped') into two parts such that D_1 is true in the first part and D_2 is true in the second part.
- $(\rho, [b, e]) \models_{RDC} D_1 \wedge D_2$ iff $(\rho, [b, e]) \models_{RDC} D_1$ and $(\rho, [b, e]) \models_{RDC} D_1$.
- $(\rho, [b, e]) \models_{RDC} \neg D$ iff not $(\rho, [b, e]) \models_{RDC} D$.

Let us fix two individual variables t and t'. Our translation Tr of RDC into monadic logic is parameterized by these two variables. Let $D(x_1, \ldots x_n)$ be a RDC formula the free variables $\{x_1 \ldots x_n\}$. Our translation will map $D(x_1, \ldots x_n)$ into $L_2^<$ formula $D'(x_1, \ldots x_n, t, t')$ with free second-order variables $x_1, \ldots x_n$ and free individual variables t, t'. The following theorem will hold for the translation Tr presented below.

THEOREM 4.3
Let D be a RDC formula. Then $(\rho, [b, e]) \models_{RDC} D$ if and only if $(\rho, [t \to r, t' \to e]) \models Tr(D)$, where $[t \to b, t' \to e]$ is the environment that assigns to the first-order variables t and t' real numbers b and e.

The translation Tr is defined as follows:

State expressions: The translation of a Boolean combination S of state variables $x_1 \dots x_n$ is the formula S', obtained from S by simultaneous substitution of $x_i(t)$ for x_i; hence, $Tr(S)$ has the second-order variables $x_1, \dots x_n$ and only one free individual variable t.

Note that a Boolean combination of finitely variable signals is a finitely variable signal. Therefore, $[|S|]\rho$ can change only a finite number of times in a bounded interval. Therefore, $[|S|]\rho$ is true almost everywhere in $[b,\ e]$ iff for every m in the interval (b, e) the signal defined by the state expression S holds in some open subinterval with m as its right endpoint and in some open subinterval with m as its left endpoint.

Hence, the translation of $\lceil S \rceil$ is defined as

$$\begin{array}{ll} t < t' \wedge & \\ \forall t_1.\ t < t_1 < t' \rightarrow & (\exists t_2.t < t_2 < t_1 \wedge \forall t_3.t_2 < t_3 < t_1 \rightarrow Tr(S)\{t_3/t\}) \wedge \\ \forall t_1.\ t < t_1 < t' \rightarrow & (\exists t_2.t_1 < t_2 < t' \wedge \forall t_3.t_1 < t_3 < t_2 \rightarrow Tr(S)\{t_3/t\}) \end{array}$$

$Tr(D_1 \frown D_2)$ is defined as $\exists t_1.Tr(D_1)\{t_1/t'\} \wedge Tr(D_2)\{t_1/t\}$.

Propositional connectives are translated as usual. Namely, $Tr(\neg D)$ is $\neg Tr(D)$ and $Tr(D_1 \wedge D_2)$ is $Tr(D_1) \wedge Tr(D_2)$

It is easy to see that $Tr(D)$ is a first-order monadic formula and that Theorem 4.3 holds.

The extension QRDC of RDC by the quantifiers over the state variables, is obtained by adding to the syntax of RDC the clause: if D is a formula then $\exists x.D$ is a formula; the semantical clause for $\exists$-quantifier is defined precisely as for the standard second-order existential quantifier. The translation is extended to QRDC by the rule $Tr(\exists x.D)$ is $\exists x.Tr(D)$. It is immediate that Theorem 4.3 holds for QTLR.

As a consequence of Theorem 4.3 and Theorem 3.2 we obtain the following corollary.

COROLLARY 4.4
The extension of RDC by the quantifiers over the state variables is decidable.

4.3 Propositional mean value calculus

The set of propositional mean value calculus formulas is defined precisely as the set of RDC formulas, except that formulas of the form $\lceil S \rceil$ are replaced by the formulas $\lceil S \rceil^0$.

$(\rho,\ [b,\ e]) \models_{MVC} \lceil S \rceil^0$ iff $b = e$ and the value of the signal assigned to S in the environment ρ is $TRUE$ at the time moment e.

We translate $\lceil S \rceil^0$ as $t = t' \wedge Tr(S)$.

The semantics of chop and of the propositional connectives is defined precisely as in RDC. Also for chop and for the propositional connectives the translation is defined precisely as the translation of the corresponding RDC constructs.

It is easy to see that Tr maps PMVC formulas into first-order monadic formulas and the following theorems hold.

THEOREM 4.5
Let D be a PMVC formula. Then $(\rho,\ [b,\ e]) \models_{MVC} D$ if and only if $(\rho,\ [t \rightarrow r,\ t' \rightarrow e]) \models Tr(D)$, where $[t \rightarrow b,\ t' \rightarrow e]$ is the environment that assigns to the first-order variables t and t' real numbers b and e.

COROLLARY 4.6
The extension of PMVC by the quantifiers over the state variables is decidable.

Pandya [11] considered extensions of PMVC by other modalities. He provided a compositional translation of these extensions into monadic logic. Note that finite variability of signals was not assumed in [11].

5 Conclusion and related results

In this paper we have shown that monadic second-order logic over signals is decidable. We also illustrated the expressive power of this logic by providing compositional translations of three typical formalisms for the specification of continuous time behaviour, the temporal logic of reals [8], the restricted duration calculus and the propositional mean value calculus into first-order fragment of monadic logic. Moreover, these translations are almost direct reformulations of the semantical definitions of these formalisms. Hence, we immediately obtained the decidability of the above formalisms. Moreover, this proof can be immediately adapted to show that the extensions of TLR, of RDC and of PMVC by the quantifiers are decidable.

The time complexity of all three translations is linear and hence the deciding satisfiability/validity of TLR, RDC, PMVC formulas is linear time reducible to the problem of deciding satisfiability/validity of monadic first-order logic of order in the finitely variable structures. The space complexity of this problem is non-elementary [10, 17], i.e. there is no k such that the satisfiability of first-order monadic formulas of size n can be decided in space $\exp_k(n)$ where $\exp_k(n)$ is the k-times iterated exponential function (e.g. $exp_2(n) = 2^{2^n}$). The complexity of the satisfiability problems for RDC and PMVC is also non-elementary [16]. In [8] the complexity of the decision algorithm for TLR was not analysed. Most probably the algorithm has exponential time complexity and the satisfiability of TLR is PSPACE complete.

There exists a natural one–one correspondence between the set of signals over the alphabet $\{0,1\}^n$ and the set of second-order environments for variables $\{x_1,\ldots,x_n\}$. With a formula $\psi(x_1,\ldots,x_n)$ the set of signals which satisfies ψ through this correspondence can be associated. Such a set of signals is called the signal language definable by ψ. Note that our proof of Theorem 3.2 does not provide any information about signal languages that can be defined in monadic logic.

Let us comment about metrical extensions of monadic logic of order. In the literature instead of $L_2^<$ the language of monadic second-order theory of one successor (denoted as S1S) is often considered. The language of S1S is obtained by extending $L_2^<$ by the function $\lambda t.t+1$. For the structure ω of natural numbers (see Section 2.3.1), the successor function $\lambda t.t+1$ is definable in $L_2^<$ hence these two languages are equivalent. For continuous structures, S1S is more expressive than $L_2^<$. It is easy to show that the validity of S1S is undecidable for the signal structure.

A signal has a variability k if it does not change more than k times in any interval of length 1. A signal has bounded variability if for a natural number k it has variability k. Wilke [19] has shown that for any fixed k, the validity problem of S1S for signals of variability k is decidable. One can show that even for the first-order fragment of S1S the validity problem is undecidable for signals with bounded variability.

The full duration calculus, the mean value calculus and the temporal logic of reals allow one to specify metrical properties. All these formalisms are undecidable. It is very important to find decidable fragments of monadic logic or of these formalisms which still allow to specify a wide spectrum of 'natural' problems that arise in practice.

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