Decidable Extensions of Church's Problem

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Abstract. For a two-variable formula B(X,Y) of Monadic Logic of Order (MLO) the Church Synthesis Problem concerns the existence and construction of a finite-state operator Y=F(X) such that B(X,F(X)) is universally valid over Nat.
Büchi and Landweber (1969) proved that the Church synthesis problem is decidable.
We investigate a parameterized version of the Church synthesis problem. In this extended version a formula B and a finite-state operator F might contain as a parameter a unary predicate P.
A large class of predicates P is exhibited such that the Church problem with the parameter P is decidable.
Our proofs use Composition Method and game theoretical techniques.

1 Introduction

Two fundamental results of classical automata theory are decidability of the monadic second-order logic of order (MLO) over $\omega = (\mathbb{N}, <)$ and computability of the Church synthesis problem. These results have provided the underlying mathematical framework for the development of formalisms for the description of interactive systems and their desired properties, the algorithmic verification and the automatic synthesis of correct implementations from logical specifications, and advanced algorithmic techniques that are now embodied in industrial tools for verification and validation.

Decidable Expansions of $\boldsymbol{\omega}$ Büchi [1] proved that the monadic theory of $\omega = (\mathbb{N}, <)$ is decidable. Even before the decidability of the monadic theory of ω has been proved, it was shown that the expansions of ω by "interesting" functions have undecidable monadic theory. In particular, the monadic theory of $(\mathbb{N}, <, +)$ and the monadic theory of $(\mathbb{N}, <, \lambda x.2 \times x)$ are undecidable [15, 20]. Therefore, most efforts to find decidable expansions of ω deal with expansions of ω by monadic predicates.

Elgot and Rabin [5] found many interesting predicates P for which MLO over $(\mathbb{N}, <, P)$ is decidable. Among these predicates are the set of factorial numbers $\{n! \mid n \in \mathbb{N}\}$, the sets of k-th powers $\{n^k \mid n \in \mathbb{N}\}$ and the sets $\{k^n \mid n \in \mathbb{N}\}$ (for $k \in \mathbb{N}$).

The Elgot and Rabin method has been generalized and sharpened over the years and their results were extended to a variety of unary predicates (see e.g., [18, 16, 3]). In [11, 14] we provided necessary and sufficient conditions for the decidability of monadic (second-order) theory of expansions of the linear order of the naturals ω by unary predicates.

Church's Problem What is known as the "Church synthesis problem" was first posed by A. Church in [4] for the case of $(\omega, <)$. The Church problem is much more complicated than the decidability problem for MLO. Church uses the language of automata theory. It was McNaughton (see [9]) who first observed that the Church problem can be equivalently phrased in game-theoretic language and in recent years many authors took up the generalizations of such games for various applications of the algorithmic theory of infinite games (see e.g., [6, 10]). McNaughton considered games over ω. We consider such games over expansions of ω by unary predicates.

Let $\mathcal{M} = (\mathbb{N}, <, P)$ be the expansion of ω by a unary predicate P. Let $\varphi(X_1, X_2, Z)$ be a formula, where X_1, X_2 and Z are set (monadic predicate) variables. The *McNaughton game* $\mathcal{G}^{\mathcal{M}}_{\varphi}$ is defined as follows.

1. The game is played by two players, called Player I and Player II.
2. A *play* of the game has ω rounds.
3. At round $i \in \mathbb{N}$: first, Player I chooses $\rho_{X_1}(i) \in \{0, 1\}$; then, Player II chooses $\rho_{X_2}(i) \in \{0, 1\}$. Both players can observe whether $i \in P$.
4. By the end of the play two predicates $\rho_{X_1}, \rho_{X_2} \subseteq \mathbb{N}$ have been constructed[1]
5. Then, Player I wins the play if $\mathcal{M} \models \varphi(\rho_{X_1}, \rho_{X_2}, P)$; otherwise, Player II wins the play.

What we want to know is: Does either one of the players have a *winning strategy* in $\mathcal{G}^{\mathcal{M}}_{\varphi}$? If so, which one? That is, can Player I choose his moves so that, whatever way Player II responds we have $\varphi(\rho_{X_1}, \rho_{X_2}, P)$? Or can Player II respond to Player I's moves in a way that ensures the opposite?

At round i, Player I has access only to $\rho_{X_1}(0) \dots \rho_{X_1}(i-1)$, $\rho_{X_2}(0) \dots \rho_{X_2}(i-1)$ and $P(0) \dots P(i)$.

Hence, a strategy of Player I can be defined as a function which assigns to any finite sequence

$$(\rho_{X_1}(0), \rho_{X_2}(0), P(0)) \dots (\rho_{X_1}(i-1), \rho_{X_2}(i-1), P(i-1)) \, (*, *, P(i))$$

a value in $\{0, 1\}$ which is taken to be $\rho_{X_1}(i)$.

At round i, Player II has access only to $\rho_{X_1}(0) \dots \rho_{X_1}(i)$, $\rho_{X_2}(0) \dots \rho_{X_2}(i-1)$ and $P(0) \dots P(i)$.

Hence, a strategy of Player II can be defined as a function which assigns to any finite sequence

$$(\rho_{X_1}(0), \rho_{X_2}(0), P(0)) \dots (\rho_{X_1}(i-1), \rho_{X_2}(i-1), P(i-1)) \, (\rho_{X_1}(i), *, P(i))$$

a value in $\{0, 1\}$ which is taken to be $\rho_{X_2}(i)$.

Since strategies are functions from finite strings (over a finite alphabet) to $\{0, 1\}$ we can classify them according to their complexity. The recursive strategies, the finite-memory strategies, i.e., the strategies computable by finite-state transducers are defined in a natural way (see Sect. 3).

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[1] We identify monadic predicates with their characteristic functions.

We investigate the following parameterized version of the Church synthesis problem.

Synthesis Problems for $\mathcal{M} = (\mathbb{N}, <, P)$, where $P \subseteq \mathbb{N}$

Input: an *MLO* formula $\varphi(X_1, X_2, Z)$.
Task: Check whether Player I has a finite-memory winning strategy in $\mathcal{G}_\varphi^\mathcal{M}$ and if there is such a strategy - construct it.

To simplify notations, games and the synthesis problem were previously defined for formulas with three free variables X_1, X_2 and Z. It is easy to generalize all definitions and results to formulas $\psi(X_1, \dots, X_m, Y_1, \dots Y_n, Z_1, \dots, Z_l)$ with many variables. In this generalization at round β, Player I chooses values for $X_1(\beta), \dots, X_m(\beta)$, then Player II replies by choosing the values to $Y_1(\beta), \dots, Y_n(\beta)$ and the structure $\mathcal{M}$ provides the interpretation for $Z_1, \dots Z_l$. Note that, strictly speaking, the input to the synthesis problem is not only a formula, but a formula plus a partition of its free-variables to Player I's variables and Player II's variables and parameter's variables.

In [2], Büchi and Landweber prove the computability of the synthesis problem in $\omega = (\mathbb{N}, <)$ (no parameters).

Theorem 1.1 (Büchi-Landweber, 1969). *Let $\varphi(\bar{X}, \bar{Y})$ be a formula, where $\bar{X}$ and $\bar{Y}$ are disjoint lists of variables. Then:*
Determinacy: *One of the players has a winning strategy in the game $\mathcal{G}_\varphi^\omega$.*
Decidability: *It is decidable* which *of the players has a winning strategy.*
Finite-state strategy: *The player who has a winning strategy, also has a finite-state winning strategy.*
Synthesis algorithm: *We can compute for the winning player in $\mathcal{G}_\varphi^\omega$ a finite-state winning strategy.*

The determinacy part of the theorem follows from the topological arguments. In particular for every expansion $\mathcal{M}$ of ω by unary predicates, the game $\mathcal{G}_\varphi^\mathcal{M}$ is determinate.

Let $\mathcal{M}$ be an expansion of ω by unary predicates. We proved in [12], that there is an algorithm which for every *MLO* formula φ decides who wins $\mathcal{G}_\varphi^\mathcal{M}$ if and only if the monadic theory of $\mathcal{M}$ is decidable. Moreover, we proved that if the monadic theory of $\mathcal{M}$ is decidable, then the player who has a winning strategy in $\mathcal{G}_\varphi^\mathcal{M}$ has a recursive *MLO*-definable winning strategy which is computable from φ.

The finite-state strategy part of Theorem 1.1 fails for decidable expansions of ω. For example, let $\mathbf{Fac} = \{n! \mid n \in \mathbb{N}\}$ be the set of factorial numbers. The monadic theory of $\mathcal{M}_{fac} := (\mathbb{N}, <, \mathbf{Fac})$ is decidable by [5]. Let $\varphi(X_1, X_2, Z)$ be a formula which specifies that $t \in X_1$ iff $t+1 \in Z$ (hence for the game $\mathcal{G}_\varphi^{\mathcal{M}_{fac}}$ the moves of Player II are irrelevant). It is easy to see that Player I has a winning strategy in $\mathcal{G}_\varphi^{\mathcal{M}_{fac}}$, yet Player I has no finite-state winning strategy in this game. The results of this paper imply that the synthesis problem for $(\mathbb{N}, <, \mathbf{Fac})$ is decidable.

Main Result Our main result describes a large class of predicates P such that the synthesis problem for $(\mathbb{N}, <, P)$ is decidable.

An ω-sequence a_i is said to be ultimately periodic with lag l and period d if $a_i = a_{i+d}$ for $i > l$.

Definition 1.2. *Let $\bar{k} = (k_1 < k_2 < \dots k_i < \dots)$ be an increasing ω-sequence of integers.*

1. *$\bar{k}$ is* sparse *if for each d there is n such that $k_{i+1} - k_i > d$ for each $i > n$.* *$\bar{k}$ is* effectively sparse *if there is an algorithm that for each d computes n such that $k_{i+1} - k_i > d$ for each $i > n$.*
2. *$\bar{k}$ is* ultimately reducible *if for every $m > 1$ the sequence k_i mod m is ultimately periodic. $\bar{k}$ is* effectively ultimately reducible *if there is an algorithm that for each m computes a lag and a period of k_i mod m.*

Definition 1.3. *Let ER be the class of increasing recursive ω-sequences of integers which are effectively sparse and effectively ultimately reducible.*

Let $P \subseteq \mathbb{N}$ be a predicate. We denote by $Enum(P)$ the sequence $(k_1, k_2 \dots k_i \dots)$ which enumerates the elements of P in the increasing order. Often we do not distinguish between P and $Enum(P)$, In particular we say that a predicate is ER predicate if $Enum(P)$ is in ER. The class ER contains many interesting predicates. It contains the set Fact=$\{n! \mid n \in \mathbb{N}\}$ of factorial numbers, the sets $\{k^n \mid n \in \mathbb{N}\}$, the sets $\{n^k \mid n \in \mathbb{N}\}$. It has nice closure properties, e.g. if $\bar{k}$ and $\bar{l}$ are in ER then $\{k_i + l_i \mid i \in \mathbb{N}\}$, $\{k_i \times l_i \mid i \in \mathbb{N}\}$, and $\{k_i^{l_i} \mid i \in \mathbb{N}\}$ are in ER.

In [18], Siefkes introduced ER predicates and generalized Elgot-Rabin contraction method to prove that for every ER predicate P the monadic theory of $\mathcal{M} = (\mathbb{N}, <, P)$ is decidable. Our main results show that the synthesis problem for each predicate $P \in ER$ is decidable.

Theorem 1.4 (Main). *Let P be an ER predicate and let $\mathcal{M} = (\mathbb{N}, <, P)$. There is an algorithm that for every MLO formula $\varphi(X_1, X_2, Z)$ decides whether Player I has a finite-memory winning strategy in $\mathcal{G}_\varphi^{\mathcal{M}}$, and if so constructs such a strategy.*

Our algorithm is based on game theoretical techniques and the composition method developed by Feferman-Vaught, Shelah and others.

Organization of the paper The article is organized as follows. The next section recalls standard definitions about the monadic second-order logic of order, and summarizes elements of the composition method. In Section 3, we introduce game-types, define games on game types and show that these game are reducible to the McNaughton games. Section 4 consider games over finite chains. Sufficient conditions are provided for existence of a finite state strategies which uniformly wins over a class of finite chains.

Section 5 describes an algorithm for the synthesis problem over the expansions of ω by ER predicates, and proves the soundness of the algorithm, i.e., if the algorithm outputs a strategy for $\mathcal{G}_\varphi^{\mathcal{M}}$, then it is a finite state strategy which wins φ over $\mathcal{M}$. The proof of completeness appears in the full version of this paper [13]. Further results and open questions are discussed in Sect. 6.

2 Preliminaries and Background

We use i, j, n, k, l, m, p, q for natural numbers. We use $\mathbb{N}$ for the set of natural numbers and ω for the first infinite ordinal. We use the expressions "*chain*" and "*linear order*" interchangeably. A chain with m elements will be denoted by m.

We use $\mathbb{P}(A)$ for the set of subsets of A.

2.1 The Monadic Logic of Order (*MLO*)

Syntax The syntax of the monadic second-order logic of order - *MLO* has in its vocabulary *individual* (first order) variables $t_1, t_2 \ldots$, monadic *second-order* variables $X_1, X_2 \ldots$ and one binary relation $<$ (the order).

Atomic formulas are of the form $X(t)$ and $t_1 < t_2$. Well formed formulas of the monadic logic *MLO* are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$ and the first-order quantifiers $\exists t$ and $\forall t$, and the second-order quantifiers $\exists X$ and $\forall X$. The quantifier depth of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

We use upper case letters X, Y, Z,... to denote second-order variables; with an overline, $\bar{X}$, $\bar{Y}$, etc., to denote finite tuples of variables.

Semantics A *structure* is a tuple $\mathcal{M} := (A, <^{\mathcal{M}}, \bar{P}^{\mathcal{M}})$ where: A is a non-empty set, $<^{\mathcal{M}}$ is a binary relation on A, and $\bar{P}^{\mathcal{M}} := \left(P_1^{\mathcal{M}}, \ldots, P_l^{\mathcal{M}}\right)$ is a *finite* tuple of sub*sets* of A.

If $\bar{P}^{\mathcal{M}}$ is a tuple of l sets, we call $\mathcal{M}$ an l*-structure*. If $<^{\mathcal{M}}$ linearly orders A, we call $\mathcal{M}$ an l*-chain*. When the specific l is unimportant, we simply say that $\mathcal{M}$ is a *labeled* chain.

Suppose $\mathcal{M}$ is an l-structure and φ a formula with free-variables among $X_1, \ldots, X_l$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as usual, understanding that the second-order quantifiers range over *subsets* of A.

Let $\mathcal{M}$ be an l-structure. The *monadic theory* of $\mathcal{M}$, $MTh(\mathcal{M})$, is the set of all formulas with free-variables among $X_1, \ldots, X_l$ satisfied by $\mathcal{M}$.

From now on, we omit the superscript in '$<^{\mathcal{M}}$' and '$\bar{P}^{\mathcal{M}}$'. We often write $(A, <) \models \varphi(\bar{P})$ meaning $(A, <, \bar{P}) \models \varphi$.

For a chain $\mathcal{M} = (A, <, \bar{P})$ and a subset I of A, we denote by $\mathcal{M} \restriction I$ the subchain of $\mathcal{M}$ over the set I.

2.2 Elements of the composition method

Our proofs make use of the technique known as the composition method developed by Feferman-Vaught and Shelah [8, 17]. To fix notations and to aid the reader unfamiliar with this technique, we briefly review the definitions and results that we require. A more detailed presentation can be found in [19] or [7].

Let $n, l \in \mathbb{N}$. We denote by $\mathfrak{Form}_l^n$ the set of *MLO* formulas with free variables among $X_1, \ldots, X_l$ and of quantifier depth $\leq n$.

Definition 2.1. *Let $n, l \in \mathbb{N}$ and let $\mathcal{M}, \mathcal{N}$ be l-structures. The* n-theory *of $\mathcal{M}$ is $Th^n(\mathcal{M}) := \{\varphi \in \mathfrak{Form}_l^n \mid \mathcal{M} \models \varphi\}$. If $Th^n(\mathcal{M}) = Th^n(\mathcal{N})$, we say that $\mathcal{M}$ and $\mathcal{N}$ are* n-equivalent *and write $\mathcal{M} \equiv^n \mathcal{N}$.*

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and $l > 0$, the set $\mathfrak{Form}_l^n$ is infinite. However, it contains only finitely many semantically distinct formulas. So, there are finitely many $\equiv^n$-equivalence classes of l-structures. In fact, we can compute characteristic formulas for the $\equiv^n$-equivalence classes:

Lemma 2.2 (Hintikka Lemma). *For $n, l \in \mathbb{N}$, we can compute a* finite *set $Char_l^n \subseteq \mathfrak{Form}_l^n$ such that:*

- *For every $\equiv^n$-equivalence class C there is a* unique *$\tau \in Char_l^n$ such that for every l-structure $\mathcal{M}$: $\mathcal{M} \in C$ iff $\mathcal{M} \models \tau$.*
- *Every MLO formula $\varphi(X_1, \dots X_l)$ with $\mathrm{qd}(\varphi) \leq n$ is equivalent to a (finite) disjunction of characteristic formulas from $Char_l^n$. Moreover, there is an algorithm which for every formula $\varphi(X_1, \dots X_l)$ computes a finite set $G \subseteq Char_l^{\mathrm{qd}(\varphi)}$ of characteristic formulas, such that φ is equivalent to the disjunction of all the formulas from G.*

*Any member of $Char_l^n$ we call a (n, l)-*Hintikka formula *or (n, l)-characteristic formula. We use τ, τ_i, τ^j to range over the characteristic formulas and G, G_i, G' to range over sets of characteristic formulas.*

Definition 2.3 (n-Type). *For $n, l \in \mathbb{N}$ and an l-structure $\mathcal{M}$, we denote by $type_n(\mathcal{M})$ the unique member of $Char_l^n$ satisfied by $\mathcal{M}$ and call it the n-*type *of $\mathcal{M}$.*

Thus, $type_n(\mathcal{M})$ determines $Th^n(\mathcal{M})$ and, indeed, $Th^n(\mathcal{M})$ is computable from $type_n(\mathcal{M})$.

Definition 2.4 (Sum of chains). (1) *Let $l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a chain and $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ a sequence of l-chains. Write $\mathcal{M}_\alpha := (A_\alpha, <^\alpha, {P_1}^\alpha, \dots, {P_l}^\alpha)$ and assume $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$ are in I. The* ordered sum *of $\mathfrak{S}$ is the l-chain*

$$\sum_{\mathcal{I}} \mathfrak{S} := (\bigcup_{\alpha \in I} A_\alpha, <^{\mathcal{I},\mathfrak{S}}, \bigcup\nolimits_{\alpha \in I} {P_1}^\alpha, \dots, \bigcup_{\alpha \in I} {P_l}^\alpha), \text{ where}$$

if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathcal{I},\mathfrak{S}} a$ iff $\beta <^{\mathcal{I}} \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathcal{M}_\alpha$'s are not disjoint, replace them with isomorphic l-chains that have disjoint domains, and proceed as before.

(2) *If for all $\alpha \in I$, $\mathcal{M}_\alpha$ is isomorphic to $\mathcal{M}$ for some* fixed *$\mathcal{M}$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M} \times \mathcal{I}$.*

(3) *If $\mathcal{I} = (\{0, 1\}, <)$ and $\mathfrak{S} = (\mathcal{M}_0, \mathcal{M}_1)$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M}_0 + \mathcal{M}_1$.*

We will use only special cases of this definition in which the index chain $\mathcal{I}$ and the summand chains $\mathcal{M}_\alpha$ are finite or of the order type ω.

The next proposition says that taking ordered sums preserves $\equiv^n$-equivalence.

Proposition 2.5. *Let $n, l \in \mathbb{N}$. Assume:*

1. *$(I, <^{\mathcal{I}})$ is a linear order,*

2. $\left(\mathcal{M}^0_\alpha \mid \alpha \in I\right)$ *and* $\left(\mathcal{M}^1_\alpha \mid \alpha \in I\right)$ *are sequences of* l*-chains, and*
3. for every $\alpha \in I$, $\mathcal{M}^0_\alpha \equiv^n \mathcal{M}^1_\alpha$.

Then, $\sum_{\alpha\in I} \mathcal{M}^0_\alpha \equiv^n \sum_{\alpha\in I} \mathcal{M}^1_\alpha$.

This allows us to define the sum of formulas in $Char^n_l$ with respect to any linear order.

Definition 2.6. (1) *Let* $n, l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ *a chain,* $\mathfrak{H} := (\tau_\alpha \mid \alpha \in I)$ *a sequence of* (n, l)*-Hintikka formulas. The* ordered sum *of* $\mathfrak{H}$, *(notations* $\sum_{\mathcal{I}} \mathfrak{H}$ *or* $\sum_{\alpha\in\mathcal{I}} \tau_\alpha$*), is* an *element* τ *of* $Char^n_l$ *such that:*

if $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ *is a sequence of* l*-chains and* $type_n(\mathcal{M}_\alpha) = \tau_\alpha$ *for* $\alpha \in I$, *then*

$$type_n(\sum_{\mathcal{I}} \mathfrak{S}) = \tau.$$

(2) *If for all* $\alpha \in I$, $\tau_\alpha = \tau$ *for some* fixed $\tau \in Char^n_l$, *we denote* $\sum_{\alpha\in\mathcal{I}} \tau_\alpha$ *by* $\tau \times \mathcal{I}$.

(3) *If* $\mathcal{I} = (\{0, 1\}, <)$ *and* $\mathfrak{H} = (\tau_0, \tau_1)$, *we denote* $\sum_{\alpha\in\mathcal{I}} \tau_\alpha$ *by* $\tau_0 + \tau_1$.

The following fundamental result of Shelah can be found in [17]:

Theorem 2.7 (Addition Theorem). *The function which maps the pairs of characteristic formulas to their sum is a recursive function. Formally, the function* $\lambda n, l \in \mathbb{N}.\lambda\tau_0, \tau_1 \in Char^n_l.\tau_0 + \tau_1$ *is recursive.*

We often use the following well-known lemmas:

Lemma 2.8. *For every* $n \in \mathbb{N}$ *there is* $N_0(n)$ *such that for every sentence* φ *of the quantifier depth at most* n *and every* $m \geq N_0$*:*

φ *is satisfiable over the* m*-element chain iff it is satisfiable over the* $m + N_0$*-element chain, i.e.,* $m \equiv^n m + N_0$.

Furthermore, N_0 *is computable from* n.

Lemma 2.9. *For every* $n \in \mathbb{N}$ *there is* $N_1(n)$ *such that for every* $\mathcal{M} = (A, <, P)$*: if* $n_1 > n_2 \geq N_1$ *and* $n_1 = n_2$ *mod* N_1, *then* $\mathcal{M} \times n_1 \equiv^n \mathcal{M} \times n_2$. *Moreover,* N_1 *is computable from* n.

3 Game types

In this section we introduce game-types; their role for games is similar to the role of types for *MLO*. We define games on game types and show that these game are reducible to the McNaughton games. But first we introduce a terminology, define finite-memory strategies and fix some notational conventions.

Let $\mathcal{M} := \left(\mathbb{N}, <, \bar{P}\right)$ be an l-chain and let $\rho := (\rho_{X_1}(0), \rho_{X_2}(0)) \ldots (\rho_{X_1}(i), \rho_{X_2}(i)) \ldots$ be a play. We denote by $\mathcal{M}^\frown\rho$ the expansion of $\mathcal{M}$ by the predicates ρ_{X_1} and ρ_{X_2}. We say that the m-type of ρ is τ if $\tau = type_m(\mathcal{M}^\frown\rho)$. Whenever $\mathcal{M}$ is clear from the context we write $type_m(\rho)$ for $type_m(\mathcal{M}^\frown\rho)$.

A strategy for Player I for games over l-chains is a transducer which consists of a set Q - memory states, an initial state q_{init}, the memory update functions $\mu_1 : Q \times \{0,1\}^l \to Q$ and $\mu_2 : Q \times \{0,1\} \to Q$, and the output function $\theta : Q \to \{0,1\}$.

A strategy is finite-memory (or finite-state) if its set of memory states is finite.

During a play at round i, Player I first updates the state according to μ_1 and the values of predicates $\bar{P}(i)$, then outputs its value according to θ , and then after a move of Player II update the state according to μ_2. Hence, a play $\rho := (\rho_{X_1}(0), \rho_{X_2}(0)) \dots (\rho_{X_1}(i), \rho_{X_2}(i)) \dots$ is consistent with such a strategy if there are $q_0, q'_0 \dots q_i q'_i$ such that $q_0 = \mu_1(q_{init}, \bar{P}(0))$, $\rho_{X_1}(i) = \theta(q_i)$, $q'_i = \mu_2(q_i, \rho_{X_2}(i))$ and $q_{i+1} = \mu_1(q'_i, \bar{P}(i+1))$.

Notational Conventions

1. In Hintikka's Lemma we considered formulas with the free variables among $X_1, \dots, X_l$. It is trivially can be extended to formulas with free second-order variables in any finite list $\bar{V}$. In particular we use $Char^k(X, Y, Z)$ for the set of Hintikka formulas of quantifier depth k with free variables X, Y, Z.
2. Whenever we deal with the synthesis problem over an l-chain $\mathcal{M} = (\mathbb{N}, <, P_1, \dots, P_l)$, we will often replace variables Z_i by the predicate P_i; in particular we will write "$\varphi(X_1, X_2, P_1, \dots, P_l)$" instead of "$\varphi(X_1, X_2, Z_1, \dots, Z_l)$"
3. By Lemma 2.2, for every formula $\varphi(X_1, X_2, P)$ of a quantifier depth n there is $G \subseteq Char^n(X_1, X_2, P)$ such that φ is equivalent to the disjunction of all formulas from G. Moreover, G is computable from φ. We often identify φ with this set G and write "$\mathcal{G}_G^{\mathcal{M}}$" instead of "$\mathcal{G}_\varphi^{\mathcal{M}}$".

Definition 3.1. *Let $\mathcal{M}$ be an l-chain, st be a strategy, and $G \subseteq Char^m(X_1, X_2, \bar{P})$. st* wins *G over $\mathcal{M}$ iff the m-type of every play (on $\mathcal{M}$) consistent with st is in G.*

Definition 3.2 (Game Types). *Let $n \in \mathbb{N}$.*

Game type of a chain *Let $\mathcal{M} := (A, < \bar{P})$ be an l-chain, where $(A, <)$ is finite or of order type ω. The n-game-type of $\mathcal{M}$ is defined as:*
game-type$_n(\mathcal{M}) := \{G \subseteq Char^n(X_1, X_2, \bar{P}) \mid$ Player I wins $\mathcal{G}_G^{\mathcal{M}}\}$.

Formal game-type *A formal (n, l)-game-type is an element*[2] *of $\mathbb{P}(\mathbb{P}(Char^n(X_1, X_2, \bar{P})))$, where $\bar{P}$ is an l-tuple $(P_1, \dots, P_l)$ of variables. We denote by $Gtype_l^n$ the set of formal (n, l)-game-types.*

Let F be a function from $\mathbb{N}$ into Gtype_1^n and $G \subseteq Char^n(X_1, X_2, P)$. We consider the following ω-game $\text{Game}(F, G)$.

Game(F, G): The game has ω rounds and it is defined as follows:

- **Round i:** Player I chooses $G_i \in F(i)$. Then, Player II chooses $\tau_i \in G_i$.
- **Winning conditions:** Let τ_i $(i \in \mathbb{N})$ be the sequence of moves of Player II in the play. Player I wins the play if $\Sigma_i \tau_i \in G$.

[2] recall that $\mathbb{P}(A)$ stands for the set of subsets of A.

The following lemma is immediate:

Lemma 3.3. *if* $\forall i\big(F_1(i) \subseteq F_2(i)\big)$, $G_1 \subseteq G_2$ *and Player I wins* $Game(F_1, G_1)$, *then Player I wins* $Game(F_2, G_2)$.

The following proposition plays an important role in our proofs:

Proposition 3.4. *Assume that* $F(i)$ $(i \in \mathbb{N})$ *is ultimately periodic. Then, it is decidable which of the players wins* $Game(F, G)$*, Moreover, the winner has a finite-memory winning strategy which is computable from* G.

4 Winning strategies over classes of finite chains

In the introduction we defined McNaugton's games over expansions of ω. In this subsection we will consider the games over expansions of finite chains. These games are defined similarly. The only difference is that these games are of finite length. The games over an l-chains with m elements have m rounds.

The following lemma says that there is a sentence which uniformly expresses that Player I has a winning strategy in the game with winning condition φ.

Lemma 4.1. *For every* φ *there is a formula* $win(\varphi)$ *such that for every finite* l*-chain* $\mathcal{M}$*, Player I has a winning strategy in* $\mathcal{G}_\varphi^{\mathcal{M}}$ *iff* $\mathcal{M} \models win(\varphi)$*. Furthermore,* $win(\varphi)$ *is computable from* φ.

Proof. (Sketch) In [11] we proved much stronger result (Theorem 2.3 in [11]) which says that there is a formula win_φ such if $\mathcal{M}$ is an expansion of ω, then Player I has a winning strategy in $\mathcal{G}_\varphi^{\mathcal{M}}$ iff $\mathcal{M} \models win_\varphi$. $\square$

Recall that we identify a subset G of $Char^m(X_1, X_2, \bar{P})$ with the disjunction $\vee_{\tau \in G}\tau$. In particular, for $G \subseteq Char^m(X_1, X_2, \bar{P})$ we write $win(G)$ for $win(\vee G)$.

For $C \subseteq \mathbb{P}(Char^m(X_1, X_2, \bar{P}))$ we write $Win(C)$ for $\wedge_{G \in C} win(G)$. $Win(C)$ expresses that Player I has a winning strategy for every $G \in C$.

Definition 4.2 (Residual). *For* $\tau \in Char^m$ *and* $G \subseteq Char^m$*, define* $res_\tau(G)$ *as* $res_\tau(G) := \{\tau' \mid \tau + \tau' \in G\}$*; define* $Res(G)$ *as* $Res(G) := \{res_\tau(G) \mid \tau \in G\}$.

Assume that ρ is a partial play of type τ. Player I can win $res_\tau(G)$ after ρ iff he has a strategy which ensures that every extension of ρ wins G.

Let st be a strategy of Player I and $\mathcal{C}$ be a class of chains. We say that st wins φ over $\mathcal{C}$ iff st is a winning strategy in $\mathcal{G}_\varphi^{\mathcal{M}}$ for every $\mathcal{M} \in \mathcal{C}$.

Lemma 4.3. *Assume that* $\mathcal{M}_0$ *and* $\mathcal{M}_1$ *are finite* l*-chains. If* $\mathcal{M}_0 \models win(G)$ *and* $\mathcal{M}_1 \models Win(Res(G))$ *then Player I has a finite-memory strategy which wins* G *over the class* $\{\mathcal{M}_0 + \mathcal{M}_1 \times k \mid k \in \mathbb{N}\}$ *of* l*-chains.*

Proof. Let k_0 and k_1 be the length of $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively. Consider the following strategy of Player I:

Play first k_0 rounds according to his winning strategy for $win(G)$. For every $j \in \mathbb{N}$ if the m-type of the play after $k_0 + jk_1$ rounds is τ then play the next k_1 rounds according to the winning strategy for $win(res_\tau(G))$.

It is easy to show by the induction on j that if a play ρ is played according to this strategy, then after $k_0 + jk_1$ rounds its m-type is in G. Therefore, it is a winning strategy for Player I.

Player I needs only a finite memory to keep the information about the m-type of the play τ_i up to each round i. After a round i he should add to τ_{i-1} the type of the play during the round i, i.e., to add to τ_{i-1} the m-type of one element chain expanded by the predicates $\rho_{X_1}(i)$, $\rho_{X_1}(i)$ and $P(i)$. Player I can calculate in a finite memory whether the current round number is $k_0 + jk_1$ for some $j \in \mathbb{N}$. Hence, this strategy is a finite-memory strategy. □

Definition 4.4. *Let $\mathcal{M}$ be an l-chain, st be a strategy, and $G \subseteq Char^m(X_1, X_2, \bar{P})$. st* realizes *G on $\mathcal{M}$ if it wins $\mathcal{G}_G^{\mathcal{M}}$ and for every m-type $\tau \in G$ there is a play ρ consistent with st such that $type_m(\mathcal{M}^\frown\rho) = \tau$,*

In other words st realizes G in $\mathcal{M}$, if st wins $\mathcal{G}_G^{\mathcal{M}}$ and there is no $G_1 \subsetneq G$ such that st wins $\mathcal{G}_{G_1}^{\mathcal{M}}$. Recall that for $n \in \mathbb{N}$ we also denote by n the finite chain with n elements.

Lemma 4.5. *1. If for $n_1 < n_2$ a strategy realizes G over chains n_1 and n_2, then $Win(Res(G))$ is satisfiable over the chain $n_2 - n_1$.*
2. If for $n_1 < n_2$ a strategy realizes G over n_1 and wins G over n_2, then $Win(Res(G))$ is satisfiable over $n_2 - n_1$.

Proof. (1) follows from (2). (2) follows from the definition of Win and Definitions 4.2 and 4.4. □

Proposition 4.6. *For $m \in \mathbb{N}$, let n be an upper bound on the quantifier depth of $win(G)$ for every $G \subseteq Char_2^m$, and let N_0 be computed from n as in Lemma 2.8. For every $i \in [0, N_0 - 1)$ the following are equivalent:*

1. *Player I has a finite-memory strategy which wins G over the class $\{t > N_0 \mid t \bmod N_0 = i\}$ of finite chains.*
2. *Player I has a finite-memory strategy which wins G over an infinite subclass of $\{t > N_0 \mid t \bmod N_0 = i\}$.*
3. *There is a finite-memory strategy which realizes $G_1 \subseteq G$ over n_1 and over n_2 for some $n_2 > n_1 \geq N_0$ such that $n_1 \bmod N_0 = n_2 \bmod N_0 = i$.*
4. *There is $G_1 \subseteq G$ such that $N_0 \models win(G')$ for every $G' \in Res(G_1)$, and $N_0 + i \models win(G_1)$.*

Proof. The implication (1) ⇒ (2) is immediate.

(2) ⇒ (3). If a strategy wins G over $\mathcal{M}$ then it realizes a subset of G. Since the set of subset of G is finite, it follows that there is a subset of G which is realized infinitely often and therefore at least twice.

(3) ⇒ (4) follows from Lemmas 2.8 and 4.5.

(4) ⇒ (1) follows from Lemma 4.3. □

Proposition 4.6 is crucial for the design of our algorithm, due the decidability of (4).

5 Algorithm

In this section we describe an algorithm for the synthesis problem for the expansions of ω by ER predicates. For every *MLO* formula $\varphi(X_1, X_2, P)$, first construct a set of the characteristic formulas G such that φ is equivalent to their disjunction and then use the following algorithm.

> Synthesis algorithm over $\mathcal{M} := (\mathbb{N}, <, P)$ where P is in ER
>
> *Instance:* $m \in \mathbb{N}$.
> *Task:* Find the set $Out = \{G \subseteq Char^m(X_1, X_2, P) \mid$ Player I has a finite-memory winning strategy in $\mathcal{G}_G^{\mathcal{M}}\}$, and for each $G \in Out$ construct a finite-memory strategy $st(G)$ which wins G over $\mathcal{M}$.

We prove the soundness of the algorithm, i.e., if $G \in Out$, then there is a finite-state strategy which wins G over $\mathcal{M}$. The proof of the reverse implication appears in the full version of this paper [13].

Let us first illustrate some ideas of the algorithm for $\mathcal{M}_{ex} := (\mathbb{N}, <, P_{ex})$, where $P_{ex} := (k_l \mid l \in \mathbb{N})$ and $k_{l+1} - k_l = l!$. Let st be a finite-memory strategy. Note that there is $l_{st}(m)$ such that for every $G \subseteq Char^m$: st wins G on $\mathcal{M}_{ex} \bigm| [k_{l_{st}}, \infty)$ iff st wins G on $\mathcal{M}_{ex} \bigm| [k_l, \infty)$ for every $l \geq l_{st}$. Recall that $\mathcal{M} \bigm| I$ is the subchain of $\mathcal{M}$ over the set I.

We can compute $U_{st}^{\infty} := \{G \subseteq Char^m \mid st$ wins G on $\mathcal{M}_{ex} \bigm| [k_l, \infty)$ for every $l \geq l_{st}\}$. For $l \in \mathbb{N}$ we can compute $V_{st}^{l} := \{G \subseteq Char^m \mid st$ wins G on $\mathcal{M}_{ex} \bigm| [0, k_l)\}$ which is a periodic sequence. From U_{st}^{∞} and $\{V_{st}^{l}\}_{l=0}^{\infty}$ we can compute $Out_{st} := \{G \subseteq Char^m \mid st$ wins G on $\mathcal{M}_{ex}\}$. Of course, we could compute Out_{st} directly from the description of st. However, our algorithm computes $U := \{G \subseteq Char^m \mid$ there is a finite-memory strategy st such that $G \in U_{st}^{\infty}\}$, and the sequence $V^l := \{G \subseteq Char^m \mid$ there is a finite-memory strategy st which wins G on $\mathcal{M}_{ex} \bigm| [0, k_l)\}$. The sequence $\{V^l\}_{l=0}^{\infty}$ is periodic. From U and $\{V^l\}_{l=0}^{\infty}$ we can compute the desirable Out.

An important property of P_{ex} is that for every n, and every $l > n$ the distance between l-th and $l+1$-th elements of P_{ex} is equal modulo n to the distance between n-th and the $n+1$-th elements of P_{ex}. Usually, this property fails for *ER* predicates; however, the sequence of the distances modulo n behaves periodically. Our algorithm is more subtle than the above sketch for P_{ex} and relies on this periodicity.

Conventions. Let $\tau(X_1, X_2)$ be an m-type for $m > 0$. There is a unique m-type $\tau^*(X_1, X_2, P)$ such that $\tau \rightarrow (\tau^*(X_1, X_2, P) \wedge \forall t \neg P(t))$. We often will not distinguish between τ and the corresponding τ^*. In particular, for m-type $\tau_1(X_1, X_2, P)$ we write $\tau + \tau_1$ instead of $\tau^* + \tau_1$. We also lift this correspondence to sets of m-types;; for a set $G \subseteq Char_2^m$ we denote by G the set $G^* := \{\tau^* \mid \tau \in G\}$.

Now we are going to describe our algorithm.

Step 1

1. Compute $One := \big\{G \subseteq Char^m(X_1, X_2, P) \mid$ Player I has a strategy which wins G over one element structure $(0, <, \{0\})$.

For $G \in One$, we denote by $st_1(One, G)$ the corresponding winning strategy.

2. Let N_0 be defined from m as in Proposition 4.6. For $i = 0, \dots N_0 - 1$ compute $CWIN^i := \{G \subseteq Char^m(X_1, X_2) \mid$ Player I has a finite-memory strategy which wins G over the class $\{t > N_0 \mid t \ mod \ N_0 = i\}\}$. This set is computable by condition (4) of Proposition 4.6.

   For $G \in CWIN^i$, we denote by $st_1(i, G)$ the corresponding finite-memory winning strategy; this strategy is computable by Lemma 4.3, since the condition (4) of Proposition 4.6 holds.

Step 2 Let $\bar{k} := k_0 < k_1 < \dots < k_i < \dots$ be the enumeration of the elements of P in the increasing order. Compute l and p such that for every n greater than l:

1. $k_{n+1} - k_n > N_0$ and
2. $(k_{n+1} - k_n) \ mod \ N_0 = (k_{n+p+1} - k_{n+p}) \ mod \ N_0$
3. For $j < p$, set $d_j := k_{l+j+1} - k_{l+j} \ mod \ N_0$.

(To compute such l and p we need our assumption that $P \in ER$.)

Step 3 Let $F : \mathbb{N} \to \text{Gtype}^m(X_1, X_2, P)$ be defined as follows:

$$F(i) = \begin{cases} One & \text{if } i \text{ is even} \\ CWIN^{d_j} & \text{if } i = 2s+1 \text{ and } s \ mod \ p = j \end{cases}$$

Note that F is a periodic sequence.

Use Proposition 3.4 to compute the set $U := \{G \subseteq Char^m(X_1, X_2, P) \mid$ Player I has a finite-memory strategy which wins $Game(F, G)\}$.

For $G \in U$, we denote by $st_{main}(F, G)$ the corresponding finite-memory winning strategy.

Now, for $G \in U$ we describe a finite-memory strategy $st_3(F, G)$ which wins G over the class $\{\mathcal{M}_i := \mathcal{M} \bigm| [k_{l+pi}, \infty) \mid i \in \mathbb{N}\}$ of chains.

We organize our description of how strategy $st_3(F, G)$ behaves on $\mathcal{M}_i := \mathcal{M} \bigm| [k_{l+pi}, \infty)$ in sessions. For $s \in \mathbb{N}$, the session $2s$ is played on the one element subchain of $\mathcal{M}_i$ isomorphic to $(0, < \{0\})$; the session $2s+1$ will be played on the subchain $\mathcal{M} \bigm| (k_{l+pi+s}, k_{l+pi+s+1})$ which is isomorphic to the $(k_{l+pi+s+1} - k_{l+pi+s})$-element chain expanded by the empty predicate.

Session 0. Let G_0 be the first move of $st_{main}(F, G)$. Then Player I will move according to his winning strategy in $st_1(One, G_0)$. After a move of Player II, the m-type of the partial play ρ_0 is some $\tau_0 \in G_0$.

Session $2s+1$. Let G_{2s+1} be the move of Player I according to $st_{main}(F, G)$ after a partial play $G_0\tau_0G_1\tau_1 \dots G_{2s}\tau_{2s}$. Then Player I will play according to his strategy in $st_1(d_{(s \ mod \ p)}, G_{2s+1})$ until he reads one on P (recall that d_j, were defined in Step 2). At this point the type of a subplay ρ_{2s+1} during this round will be $\tau_{2s+1} \in G_{2s+1}$.

Session $2s$. *($s > 0$)* Let G_{2s} be the move of Player I according to $st_{main}(F, G)$ after a partial play $G_0\tau_0G_1\tau_1 \dots G_{2s-1}\tau_{2s-1}$. Player I will move according to his winning strategy in $st_1(One, G_{2s})$. After a move of Player II, the m-type of the partial play ρ_{2s} during this session will be some $\tau_{2s} \in G_{2s}$.

Observe that this is indeed a finite-memory strategy. Like in the proof of Lemma 4.3, Player I can compute in a finite memory at each session s the m-type τ_s of the subplay during session s, and then after this session to supply only this m-type to $st_{main}(F,G)$ (and not the whole history $G_0\tau_0 \dots G_s\tau_s$).

This strategy wins G because the sequence $G_0\tau_0 \dots G_s\tau_s \dots$ played over the sessions is consistent with the wining strategy $st_{main}(F,G)$ in $Game(F,G)$.

Step 4 We are going to compute the set $V := \{G \subseteq Char^m(X_1,X_2,P) \mid$ Player I has a strategy which wins G over $\mathcal{M} \upharpoonright [0,k_{l+pi})$ for some $i \in \mathbb{N})\}$.

Let n be the quantifier depth of $win(G)$.

By our choice of N_0, l and p (in Step 1 and Step 2) we know that for every i:

$$\mathcal{M} \upharpoonright [k_{l+i}, k_{l+i+1}) \equiv^n \mathcal{M} \upharpoonright [k_{l+i+p}, k_{l+i+1+p}))$$

Hence, for every i:

$$\begin{aligned} \mathcal{M} \upharpoonright [k_{l+pi}, k_{l+pi+p}) = \textstyle\sum_{s=0}^{p-1} \mathcal{M} \upharpoonright [k_{l+pi+s}, k_{l+pi+s+1}) \equiv^n \\ \equiv^n \textstyle\sum_{s=0}^{p-1} \mathcal{M} \upharpoonright [k_{l+s}, k_{l+s+1}) = \mathcal{M} \upharpoonright [k_l, k_{l+p}) \end{aligned}$$

Let $N_1 := N_1(n)$ be defined as in Lemma 2.9. From the above equivalence, Lemma 2.9 and Proposition 2.5, it follows that for every i there is $j \leq N_1$ such that

$$\mathcal{M} \upharpoonright [k_l, k_{l+pi}) \equiv^n \mathcal{M} \upharpoonright [k_l, k_{l+pj})$$

and hence, $\mathcal{M} \upharpoonright [0, k_{l+pi}) \equiv^n \mathcal{M} \upharpoonright [0, k_{l+pj})$.

Therefore, $V = \{G \subseteq Char^m(X_1,X_2,P) \mid \mathcal{M} \upharpoonright [0,k_{l+pj}) \models win(G)$ for some $j \leq N_1\}$. To compute the right hand side we need to check a finite set of games over finite chains. Hence, this is computable and therefore, V is computable.

For $G \in V$, let $l_G \leq N_1$ be such that $\mathcal{M} \upharpoonright [0, k_{l+pl_G}) \models win(G)$ and let $st_4(V,G)$ be the corresponding strategy which wins G over $\mathcal{M} \upharpoonright [0, k_{l+pl_G})$.

Step 5 Output $Out := \{G \subseteq Char^m(X,Y,P) \mid \exists G_1 \in V$ such that $res_\tau(G) \in U$ for every $\tau \in G_1\}$.

For every $G \in Out$ we describe a finite-memory strategy $st(G)$ which wins G over $\mathcal{M}$. Assume $G \in Out$ and let $G_1 \in V$ be such that $res_\tau(G) \in U$ for every $\tau \in G_1$. Since $G_1 \in V$, there is l_{G_1} and a strategy $st_4(V,G_1)$ which wins G_1 over $\mathcal{M} \upharpoonright [0, k_{l+pl_{G_1}})$.

Player I will play the first $l + p \times l_{G_1}$ rounds according to this winning strategy. Let ρ be a play according to this strategy, and let τ be its m-type and let $G_2 = res_\tau(G)$. The rest of the game Player I will play according to his finite-memory strategy $st_3(F,G_2)$ computed in the Step 3 Clearly, the described strategy is a finite-memory strategy.

The m-type of the whole play is in $\tau + G_2 = G$. Therefore, the described strategy is winning in $\mathcal{G}_G^{\mathcal{M}}$. This completes the description of our algorithm and the proof that if $G \in Out$, then Player I has a finite-memory winning strategy in $\mathcal{G}_G^{\mathcal{M}}$.

6 Further Results and Open Questions

We proved that the finite-memory synthesis problem is decidable for the expansions of ω by the predicates from ER. In [12] it was proved that the decidability of the monadic theory of $\mathcal{M}$ is equivalent to the decidability of the recursive strategy synthesis problem for $\mathcal{M}$.

The question whether the decidability of the monadic theory of $\mathcal{M}$ implies the decidability of the finite-memory synthesis problem for $\mathcal{M}$ remains open.

A natural question to consider is the synthesis problem for strategies between finite-memory and recursive ones, e.g., the strategies computable by push-down automata.

There are some minor modifications of the McNaughton games to the games with look-ahead. Let $\mathcal{M} = (\mathbb{N}, <, P)$ be the expansion of ω by a unary predicate P. Let h_1, h_2 be integers (look-ahead) of the players. Let $\varphi(X_1, X_2, Z)$ be a formula. The game $\mathcal{G}_{\varphi}^{\mathcal{M}}(h_1, h_2)$ with look-ahead h_1 for Player I and look-ahead h_2 for Player II is defined as follows. The game is played by two players and each of its plays has ω rounds.

1. At round $i \in \mathbb{N}$: first, Player I chooses $\rho_{X_1}(i) \in \{0, 1\}$; then, Player II chooses $\rho_{X_2}(i) \in \{0, 1\}$. Player I can observe whether $i + h_1 \in P$ and Player II can observe whether $i + h_2 \in P$
2. By the end of the play two predicates $\rho_{X_1}, \rho_{X_2} \subseteq \mathbb{N}$ have been constructed.
3. Then, Player I wins the play if $\mathcal{M} \models \varphi(\rho_{X_1}, \rho_{X_2}, P)$; otherwise, Player II wins the play.

The proof of the next proposition is similar to the proof of Theorem 1.4.

Proposition 6.1. *Let P be an ER predicate, and h_1, h_2 integers and let $\mathcal{M} = (\mathbb{N}, <, P)$. There is an algorithm that for every MLO formula $\varphi(X_1, X_2, Z)$ decides whether Player I has a finite-memory winning strategy in $\mathcal{G}_{\varphi}^{\mathcal{M}}(h_1, h_2)$, and if so, constructs such a strategy.*

It is easy to modify our proofs and to show that it is decidable whether Player II has a finite-memory winning strategy.

Section 1 (page 3) gives an example of the game $\mathcal{G}_{\varphi}^{\mathcal{M}_{fac}}$ where Player I has a winning strategy, yet he has no finite-memory winning strategy. Note that for this particular game, Player I has a finite-memory one-look-ahead winning strategy, i.e., he has a finite-memory winning strategy in $\mathcal{G}_{\varphi}^{\mathcal{M}_{fac}}(1, h_2)$ for every h_2.

Relying on the definability results in [12] we can prove the following Proposition.

Proposition 6.2 (Determinacy for look-ahead finite-memory strategy). *Let P be an ER predicate, and let $\mathcal{M} = (\mathbb{N}, <, P)$. For every MLO formula $\varphi(X_1, X_2, Z)$ there is h such that one of the players has a finite-memory winning strategy in $\mathcal{G}_{\varphi}^{\mathcal{M}}(h, h)$. Furthermore, there is an algorithm that computes such h and a finite-memory winning strategy for the winner in $\mathcal{G}_{\varphi}^{\mathcal{M}}(h, h)$.*

It is plausible that in our proofs the compositional methods can be hidden and a presentation can be given based on automata theoretic concepts. The logical n-types can be replaced by "n-types", using semigroups or automata rather than formulas to describe properties of words. The only place where automata based techniques might fail is in the proof of Proposition 3.4.

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