# Continuous time Temporal Logic with Counting

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**Abstract**

We add to the standard temporal logic $TL(\mathbf{U}, \mathbf{S})$ a sequence of "counting modalities": For each $n$ the modality $C_n(X)$, which says that $X$ will be true at least at $n$ points in the next unit of time, and its dual $\overleftarrow{C}_n$, which says that $X$ has happened $n$ times in the last unit of time. We show that this temporal logic is expressively complete for the metric predicate logic *Q2MLO*, which is expressive, decidable and easy to use. In particular the Pnueli modalities $P_n(X_1, \ldots, X_n)$, "there is an increasing sequence $t_1, \ldots, t_n$ of points in the unit interval ahead such that $t_i$ satisfies $X_i$", are definable in $TL(\mathbf{U}, \mathbf{S})$ with the counting modalities.

## 1 Introduction

The temporal logic that is based on the two modalities "Since" and "Until" is popular among computer scientists as a framework for reasoning about a system evolving in time. By Kamp's theorem [Kamp68] this logic has the same expressive power as the first order monadic logic of order, whether the system evolves in discrete steps or in continuous time. We will denote this logic by *TL*.

For systems evolving in discrete steps, this logic seem to supply all the expressive power needed. This is not the case for systems evolving in continuous time, as the logic cannot express properties like: "$X$ will occur soon" which can be given the precise form "$X$ will occur within one unit of time". Over the years different extensions of *TL* were suggested. Most extensively researched was $MITL$ [AH92, AFH96, Hen98]. Other logics are described in [BKP85, MP93]. We introduced the language $QTL$ (quantitative temporal logic) [HR99a, HR04, HR05], which we find natural and convenient. These extensions of *TL* have the same expressive power, which indicates that they capture a natural fragment of what can be said about a system which evolve in time. These "first generation" metric extensions of *TL* can be called *simple metric temporal logics*. More expressive logics involving second-order quantifiers and $\omega$-automata were considered in [Wilke94, HRS98] and in our work (see e.g. survey [HR04]).

A. Pnueli was probably the first person to question if these simple logics are expressive enough for our needs. The conjecture that they cannot express the property "$X$ and then $Y$ will both happen in the coming unit of time" is usually referred to as "Pnueli's conjecture".

In [HR07] we proved Pnueli's conjecture, and we showed a sequence of modalities of the type that Pnueli suggested, such that no finite set of modalities can express all of them. Specifically: For every natural $n$ we defined the "Pnueli modality" $P_n(X_1,\ldots,X_n)$, which states that there is an increasing sequence $t_1,\ldots,t_n$ of points in the unit interval ahead such that $t_i$ satisfies $X_i$. To deal with the past we define also the dual past modality, $\overleftarrow{P}_n(X_1,\ldots,X_n)$: there is a decreasing sequence $t_1,\ldots,t_n$ of points in the previous unit interval such that $t_i$ satisfies $X_i$.

This yields a sequence of temporal logics $TLP_n$, which is the standard temporal logic, with "Until" and "Since", and with the addition of the n-place modalities $P_n$ and $\overleftarrow{P}_n$. We note that $TLP_{n+1}$ is at least as strong as $TLP_n$ since $P_n(X_1,\ldots,X_n) = P_{n+1}(X_1,\ldots,X_n,True)$. We note also that $TLP_1$ is just the logic $QTL$ and it represents the simple metric logics.

We proved in [HR07] that:

- The sequence of temporal logics $TLP_n$ is strictly increasing in expressive power.
- Their union $TLP$ is not contained in any temporal logic with finitely many modalities (this statement is made precise in [HR07]).

The modalities $P_n$ are easily expressible in a small fragment of the monadic logic with order and with the +1 function. It follows that Kamp's theorem in its fullest does not extend to the metric case: There is no finite metric temporal logic that is equivalent to any monadic logic that can express all the operators $P_n$.

Is $TLP$ as expressive as needed, and if it is, how do we prove it? If not, what other modalities should be taken instead of the Pnueli modalities or in addition?

In [HR99, HR04] we defined the predicate logic *Q2MLO*. It allows mention of the metric only in the form $(\exists t)^{<t_0+m}_{>t_0+n}\varphi$, where $m$ and $n$ are integers with $m < n$, or with weak inequalities on one or both ends, provided the formula $\varphi$ has at most $t_0$ and $t$ as free variables. We found this logic very useful for the following reasons:

- It is powerful enough to subsume all the decidable temporal logics that we found in the literature. In particular Pnueli's modalities have a simple definition in this logic [HR99a].
- If we use only the quantifiers $(\exists t)^{<t_0+1}_{>t_0}\varphi$ and $(\exists t)^{<t_0}_{>t_0-1}\varphi$ (allowing only strict inequalities and simple unit intervals) the resulting sublogic $Q2MLO_0$ is as expressive as all of *Q2MLO* [HR08].
- The logic is decidable (for satisfiability and validity) [HR99a].

- *Q2MLO* can not be replaced by any temporal logic with finitely many modalities [HR07].

In this paper we prove that $TLP$ is expressively complete for *Q2MLO*. In fact a simpler infinite sequence modalities can replace the Pnueli modalities and still produce an expressively complete for *Q2MLO* temporal logic. We define the *Counting modalities*: For every $n$ the statement $C_n(X)$ says that $X$ will be true at least at $n$ points within the next unit of time. The dual modality $\overleftarrow{C}_n(X)$, says that $X$ was true at least $n$ times in the past unit of time. Hence, $C_n(X) = P_n(X, \ldots, X)$ and $\overleftarrow{C}_n(X) = \overleftarrow{P}_n(X, \ldots, X)$. We denote by $TLC$ the temporal logic with $Until$, $Since$ and all the counting modalities $C_n$ and $\overleftarrow{C}_n$.

The main theorem states that $TLC$ is expressively complete with respect to *Q2MLO*.

This general theorem proves in particular that every temporal modality that can be defined in *Q2MLO* is defined also in $TLC$. Thus for example Pnueli's modalities $P_n$ can be defined in terms of the counting modalities $C_n$.

The proof uses the composition theory for logics of order, and it is quite general: For any logic that obeys the appropriate composition rules the addition of metric quantifiers of the kind that we define does not add more expressive power than the ability to count.

The paper is divided as follows: In section 2 we recall the definitions and the previous results which are needed. In section 3 we prove the main theorem. In section 4 we discuss the applicability of our methods to obtain similar results in a more general context.

# 2 Monadic Logic and Quantitative Temporal Logic

## 2.1 FOMLO - First-Order Monadic Logic of Order

We recall some definitions:

**The monadic predicate logic of order - FOMLO** has in its vocabulary *individual* (first order) variables $t_0, t_1 \ldots$ and monadic *predicate* variables $X_0, X_1 \ldots$ finitely or infinitely many monadic *predicate* names $S_0, S_1, , \ldots,$ and one binary relation $<$ (the order).

The first order predicate language over this vocabulary is referred here as the **First-Order Monadic Logic of Order (*FOMLO*)**.

**A structure for** *FOMLO* is a tuple $M = \langle \mathbb{R}, <, S_1, \ldots, S_n \rangle$, where $\mathbb{R}$ is the real line, or the non negative segment of the real line, and $S_1, \ldots, S_n$ are one-place predicates (sets) that correspond to the predicate names in the logic.

When the free variables of a formula $\varphi$ are among $t_1, \ldots, t_k,\ X_1, \ldots, X_m$ and if $\tau_1, \ldots, \tau_k$ are elements of $M$ and $P_1, \ldots, P_m$ are monadic predicates on the domain of $M$, we will replace the exact yet tedious notation

$$M, \tau_1, \ldots, \tau_k; P_1, \ldots, P_m \models_{MLO} \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m),$$

by the notation

$$M \models_{MLO} \varphi(\tau_1, \ldots, \tau_k; P_1, \ldots, P_m),$$

**Remark 2.1 (Finite Variability)** *Often it is assumed that all the unary predicates that are involved have* Finite Variability*. This means that in any finite interval of time the predicate changes only finitely often its truth value. We are interested in first order logics, but even so a claim that involves all models in a language, like questions concerning satisfiability and validity, may have different answers if the claim is about general models with arbitrary unary predicates, or about logics with finite variability, where only predicates with finite variability are allowed. We consider general logics but everything that we do applies word for word to the finite variability interpretation.*

## 2.2 Temporal Logics

Temporal logics use logical constructs called "modalities" to create a language that is free from quantifiers:

**The syntax of the Temporal Logic** $TL(O_1^{(k_1)}, O_2^{(k_2)}, \ldots)$ has in its vocabulary *monadic predicate names* $S_1, S_2, \ldots$ and a (finite or infinite) sequence of *modality names* $O_1^{(k_1)}, O_2^{(k_2)}, \ldots$. The superscript $k_i$ denotes the intended arity of the modality and it is usually omitted. The formulas of this temporal logic are given by the grammar:

$$\varphi \ ::= True|\ S \mid \neg\varphi \mid \varphi \wedge \varphi \mid O^{(k)}(\varphi_1, \cdots, \varphi_k)$$

A temporal logic with a finite set of modalities is called a finite temporal logic.

**A structure for Temporal Logic**, in this work, is the non negative real line with monadic predicates $M = \langle \mathbb{R}, <, S_1, S_2, \ldots \rangle$, where the predicate $S_i$ are those which are mentioned in the formulas of the logic. Every modality $O^{(k)}$ is interpreted in the structure $M$ as an operator $O_M^{(k)} : [\mathbb{P}(\mathbb{R})]^k \to \mathbb{P}(\mathbb{R})$ which assigns "the set of points where $O^{(k)}[A_1, \ldots, A_k]$ holds" to the $k$-tuple $\langle A_1, \ldots, A_k \rangle \in \mathbb{P}(\mathbb{R})^k$. ( $\mathbb{P}(\mathbb{R})$ denotes the set of all subsets of $\mathbb{R}$). Once every modality corresponds to an operator the semantics is defined by structural induction:

- for atomic formulas: $\langle M, t \rangle \models_{TL} S \quad \text{iff} \quad t \in S$.
- for Boolean combinations the definition is the usual one.
- for $O^{(k)}(\varphi_1, \cdots, \varphi_k)$

  $$\langle M, t \rangle \models_{TL} O^{(k)}(\varphi_1, \cdots, \varphi_k) \quad \text{iff} \quad t \in O_M^{(k)}(A_{\varphi_1}, \cdots, A_{\varphi_k})$$

  where $A_\varphi = \{ \tau : \langle M, \tau \rangle \models_{TL} \varphi \}$ (we suppressed predicate names that may occur in the formulas).

For the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets $A_{\varphi_i}$ of points satisfying $\varphi_i$ and the set of points $O[A_{\varphi_1}, \ldots, A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

**Truth Tables**: A formula $\overline{O}(t, X_1, \ldots X_k)$ in the predicate logic $L$ is a *Truth Table* for the modality $O^{(k)}$ if for every structure $M$

$$O_M(A_1, \ldots, A_k) = \{\tau \ : \ M, \tau \models_{MLO} \overline{O}[\tau, A_1, \ldots, A_k]\} .$$

The modalities *until* and *since* are most commonly used in temporal logic for computer science. They are defined through the following truth tables:

- The modality $X\mathbf{U}\, Y$, "$X$ *until* $Y$", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 < t_1 \wedge Y(t_1) \wedge \forall t(t_0 < t < t_1 \rightarrow X(t))).$$

- The modality $X\mathbf{S}\, Y$, "$X$ *since* $Y$", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 > t_1 \wedge Y(t_1) \wedge \forall t(t_1 < t < t_0 \rightarrow X(t))).$$

We recall the terminology that is used when comparing the expressive power of languages.

Let $\mathcal{M}$ be a class of structures (possibly with only one structure), let $L$ be fragment of a predicate logic and let $L'$ be a temporal logic. $L$ and $L'$ are **expressively equivalent** over $\mathcal{M}$ if

1. For every formula $\varphi$ of $L'$ there is a formula $\psi(t)$ in $L$ with a single free variable, such that for every structure $M$ in $\mathcal{M}$ and for every $\tau \in M$

$$\langle M, \tau\rangle \models_{TL} \varphi \quad \text{iff} \quad M, \tau \models \psi(t)$$

2. For every formula $\psi(t)$ of $L$ there is a formula $\varphi$ in $L'$, such that for every structure $M$ in $\mathcal{M}$ and for every $\tau \in M$

$$\langle M, \tau\rangle \models_{TL} \varphi \quad \text{iff} \quad M, \tau \models \psi(t)$$

If a temporal logic $L'$ is equivalent over $\mathcal{M}$ to a predicate logic $L$ then we say that $L'$ is **expressively complete** for $L$ over $\mathcal{M}$.

Since the modalities "until" and "since" have truth tables in first-order *MLO*, the temporal logic $TL(\mathbf{U}, \mathbf{S})$ satisfies the first condition and therefore it corresponds to a fragment of first order *MLO*.

The two modalities $\mathbf{U}$ and $\mathbf{S}$ are also enough to express all the formulas of first-order *MLO* (*FOMLO*) with one free variable, so that $TL(\mathbf{U}, \mathbf{S})$ is expressively equivalent to first order *MLO*:

**Theorem 2.2** *([Kamp68, GPSS80]) The temporal logic $TL(\mathbf{U}, \mathbf{S})$ is expressively complete for FOMLO over the two canonical structures: For every formula of FOMLO with at most one free variable, there is a formula of $TL(\mathbf{U}, \mathbf{S})$, such that the two formulas are equivalent to each other, over the positive integers (discrete time) and over the real line (continuous time).*

## 2.3 The simple metric logics: Quantitative Temporal Logic, and Quantitative Monadic Logic of Order

The logics *FOMLO* and $TL(\mathbf{U}, \mathbf{S})$ are not suitable to deal with statements like "$X$ will occur within one unit of time". For the last 20 years languages that can express such properties were developed and investigated ([BKP85, MP93, GHR94] [Wilke94, Hen98, HR99a, HR05]), and most well-known, $MITL$ [AH92, AFH96, Hen98]. We will use as a framework the *Quantitative Temporal Logic,* $QTL$ which was introduced in [HR99, HR99a, HR04]. All these logics are basically equivalent [RSH98, HR99a]. $QTL$ is defined as follows:

**Definition 2.3 (Quantitative Temporal Logic)** *$QTL$,* quantitative temporal logic *is the logic $TL(\mathbf{U}, \mathbf{S})$ enhanced by the two modalities: $\diamondsuit_1 X$ and $\overleftarrow{\diamondsuit}_1 X$. These modalities are defined by the tables with free variable $t_0$:*

$$\diamondsuit_1 X: \qquad \exists t((t_0 < t < t_0 + 1) \wedge X(t))$$

$$\overleftarrow{\diamondsuit}_1 X: \qquad \exists t((t_0 - 1 < t < t_0) \wedge X(t)) \ .$$

The temporal logic $QTL$ is complete for a natural fragment of the monadic logic of order, enriched with the $+1$ function:

**Definition 2.4 (Quantitative Monadic Logic of Order)** *QMLO ,* quantitative monadic logic of order *is the predicate logic that has atomic formulas $t = s$, $t < s$ and $X(t)$, is closed under Boolean connectors and first order quantifications,* **and** *under the "metric quantifiers":*

*If $\varphi(t)$ is a formula in QMLO with $t$ its only free variable then $(\exists t)^{<t_0+1}_{>t_0}\varphi(t)$ and $(\exists t)^{<t_0}_{>t_0-1}\varphi(t)$ are formulas of QMLO (in the free variable $t_0$).*

**Theorem 2.5** *[HR99a, HR05] The temporal logic QTL is expressively complete for QMLO over the real line, or over the positive half real line.*

## 2.4 The limited expressive power of the simple metric logics

There was no reason to believe that the simple metric logics like $QTL$ have comprehensive expressive power. A. Pnueli raised this question, and he conjectured that the modality $P_2(X, Y)$ is not expressible in $MITL$, where $P_2(X, Y)$ says that $X$ and then $Y$ will be true at points in the next unit of time.

In [HR07] we proved Pnueli's conjecture, and we strengthened it significantly. To do this we defined for every natural number $n$ the "*Pnueli modality*" $P_n(X_1, \ldots, X_n)$, which states that there is an increasing sequence $t_1, \ldots, t_n$ of points in the open unit interval ahead such that $t_i$ satisfies $X_i$. We also defined the weaker "*Counting modalities*" $C_n(X)$ which state that $X$ is true at least at $n$ points in the open unit interval ahead (so that $C_n(X) = P_n(X, \ldots, X)$). In [HR07] we proved that:

**Theorem 2.6** *1. QTL (or $MITL$) with the added Pnueli modalities $P_1, \ldots, P_n$ can not express the modality $C_{n+1}$*

*2. No finite temporal logic that can be defined in second order monadic logic of order extended by the $+1$ function can express on the real line all the modalities $C_n(X)$ for all natural numbers $n$.*

This means that no finite temporal logic will suffice to express everything that is of interest. We must either give up temporal logic as means for comprehensive expressive power, or allow infinitely many modalities. Our aim in this work is to show that although an infinite collection of modalities is needed it may be a very simple collection.

## 2.5 The predicate metric logic *Q2MLO*

We found the following logic *Q2MLO* natural and suitable to deal with evolving systems. It was introduced in [HR99].

**Definition 2.7** *Q2MLO is the predicate logic that has atomic formulas $t = s$, $t < s$ and $X(t)$, is closed under Boolean connectors and first order quantifications,* **and** *under the "metric quantifiers":*

*If $\varphi(t_0, t)$ is a formula in Q2MLO with $t$ and $t_0$ its only free variables and $m < n$ are integers then $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0, t)$ is a formula of Q2MLO (in the free variable $t_0$).*

What looks like a minor difference between *QMLO* and *Q2MLO* is an essential difference. *QMLO* allows us to say that within one unit of time a punctual event will occur at some point. In *Q2MLO* we can speak about the whole interval from now up to that point. Thus each of the different Pnueli modalities has a very simple truth table in *Q2MLO* : $P_n(X_1, \ldots, X_n)$ holds at $t_0$ iff

$$(\exists t)^{<t_0+1}_{>t_0}(\exists t_1, \cdots, t_n)[(t_0 < t_1 < \cdots < t_n < t) \wedge (X_1(t_1) \wedge \cdots \wedge X_n(t_n))]$$

In [HR99, HR04, HR08] it was shown that:

**Theorem 2.8** *The validity and satisfiability problems are decidable for Q2MLO, over continuous time, whether we are interested in the class of models with finite variability, or in the class of all models.*

In [HR08] it was also shown that:

**Theorem 2.9** *Every formula of Q2MLO can be effectively replaced by an equivalent formula of Q2MLO all of whose metric quantifications are of the form $(\exists t)^{<t_0+1}_{>t_0}\varphi(t_0, t)$ and $(\exists t)^{<t_0}_{>t_0-1}\varphi(t_0, t)$.*

## 2.6 Elements of composition method

The proof of theorems 2.8 and 2.9 uses the composition method. This method is used in this paper for the proof of the main theorem. We describe the method briefly: Families of structures of a type may be combined in different ways to create a new structure of the same type. The "compositional method" applies to the case where a structure is composed from simpler structures, and the theory of the composite structure can be reduced to the theory of its components. Ehrenfeucht used it in [Ehr61] for ordered structures, and our proofs follow his steps. The method was developed and used by Feferman-Vaught [FV59], Shelah [She75] and others (see surveys [Gur85, Tho97, Mak04]). We concentrate on the case where two (or more) ordered structures are combined together to form a new ordered structure, in which all the elements of the first structure precede those of the second.

Here we need the method for a counting argument over the real line, and we will state only the simplest of the composition theorems (cf. see Lemma 9.3.2 in [GHR94]) in accordance with the way that we intend to use it in the real line.

**Theorem 2.10 (Composition Theorem)** *For every formula $\varphi(x, y)$ of first order monadic logic of order there is a finite set of pairs of formulas in the same language*

$$\langle \alpha_1(x, z), \beta_1(z, y)\rangle, \cdots, \langle \alpha_q(x, z), \beta_q(z, y)\rangle$$

*such that for every chain $M = \langle A, <, S_1, \ldots, S_k\rangle$ any three points $a < c < b$*

$$M \models \varphi(a, b) \qquad \textit{iff} \qquad M \models \bigvee_{i=1}^{q} (\alpha_i(a, c) \wedge \beta_i(c, b))$$

We are going to use this theorem when $A$ is the set of positive reals or reals and $<$ is the standard order relation on these sets.

# 3 Completeness of the counting modalities

For every $n$ the modality $C_n(X)$ is defined by the truth table in *Q2MLO*:

$$(\exists t_n)^{<t+1}_{>t}(\exists t_1 \cdots t_{n-1})[(t < t_1 < \cdots < t_n) \wedge (X(t_1) \wedge \cdots \wedge X(t_n))]$$

Similarly $\overleftarrow{C}_n(X)$ is defined by the truth table in *Q2MLO*:

$$(\exists t_n)^{<t}_{>t-1}(\exists t_1 \cdots t_{n-1})[(t > t_1 > \cdots > t_n) \wedge (X(t_1) \wedge \cdots \wedge X(t_n))]$$

We denote by $TLC$ the temporal logic $TL(\mathbf{U}, \mathbf{S})$ with all the modalities $C_n$, and $\overleftarrow{C}_n$. By a straightforward induction it follows

**Lemma 3.1** *For every formula of $TLC$ there is an equivalent formula in Q2MLO.*

The main result of the paper is the converse:

**Theorem 3.2 (Main Theorem)** *Every formula of Q2MLO with at most one free variable is effectively equivalent in the class of all structures (with or without finite variability) over $\mathbb{R}^+$ (and over $\mathbb{R}$), to a formula of $TLC$.*

The main effort will be to prove the theorem for a simple formula with a single metric quantifier. This will take up to proposition 3.8. We will then extend the result to general formulas. Since the proof is involved, we start with some notations and (ad hoc) definitions that will ease the discussion.

**Definition 3.3** *We consider formulas with at most two free variables, as specified.*

1. *A* **pure formula** *is a formula of the pure monadic logic of order (with no metric quantifiers).*

2. *A* **functional formula** *is a pure formula $\varphi(t, s)$ for which every $t$ has at most one partner $s$ that satisfies $\varphi(t, s)$. And in addition if $\varphi(t, s)$ holds then $t \leq s$.*

3. *If $\varphi(t, s)$ is functional, and $\varphi(a, b)$ holds then we say that $a$ and $b$ are* **partners** *(with respect to $\varphi$), and that the interval $[a, b]$ is* **a $\varphi$-interval**. *$b$ is the right partner of $a$ and $a$ is a left partner of $b$ . $f_\varphi(t)$ is the partial function that associates with every point its right partner, if it has one. Note that $a \leq f_\varphi(a)$, but they may be equal.*

4. *A* **simple metric formula** *is a formula of the form $(\exists s)_{>t}^{<t+1}\varphi(t, s)$, where $\varphi$ is a pure formula.*

5. *Finally a* **special formula** *has the form $(\exists s)_{\geq t}^{<t+1}\varphi(t, s)$, where $\varphi$ is functional (note that the lower inequality is weak inequality).*

To further simplify the discussion, we will avoid the careful and cumbersome distinction between the free variables of a formula $\varphi(t, s)$ and their intended interpretation in the model. We will speak freely of "the interval $(t, s)$", and say that "the interval satisfies the formula $\varphi$", instead of "the interpretation satisfies $\varphi(t, s)$".

We start with the following transformation:

**Lemma 3.4** *Every simple formula is equivalent to a special formula. Specifically:*

*Let $\varphi(t, s)$ be a first-order monadic logic of order formula. There is a functional formula $\varphi'(t, s)$ with the same free variables such that the formulas $(\exists s)_{>t}^{<t+1}\varphi(t, s)$ and $(\exists s)_{\geq t}^{<t+1}\varphi'(t, s)$ are equivalent (note the difference in the lower inequality).*

**Proof** We define first $\varphi''(t, s)$ which says that $t \leq s$ and that either $\varphi(t, s) \wedge t < s$ holds, or $s$ is a limit of a decreasing sequence of points $s_n$ for which $\varphi(t, s_n)$

holds. We will then define $\varphi'(t,s)$ to say that $s$ is the first element that satisfies $\varphi''(t,s)$.

We define:

$$Inf_{\varphi}(t,s) \equiv (\forall s' > s)(\exists s'')[(s < s'' < s') \wedge \varphi(t,s'')]$$

and

$$\varphi''(t,s) \equiv [(t < s) \wedge \varphi(t,s)] \vee [t \leq s \wedge Inf_{\varphi}(t,s)]$$

and finally

$$\varphi'(t,s) \equiv \varphi''(t,s) \wedge [\forall s'(t < s' < s) \rightarrow \neg\varphi''(t,s')]$$

We will show that $(\exists s)^{<t+1}_{>t}\varphi(t,s) \longleftrightarrow (\exists s)^{<t+1}_{\geq t}\varphi'(t,s)$ holds for every $t$. Indeed if $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ and if $s_0$ is the greatest lower bound of such elements $s$, then $t \leq s_0 < t+1$, and either $\varphi(t,s_0)$ holds, or $Inf_{\varphi}(t,s_0)$ holds. In either case $s_0$ is the first element that satisfies $\varphi''(t,s)$. On the other hand, if $(\exists s)^{<t+1}_{\geq t}\varphi'(t,s)$ is true then for some $s_0$, with $t \leq s_0 < t+1$ we have $\varphi'(t,s_0)$. This means in particular that either $(t < s_0 < t+1) \wedge \varphi(t,s_0)$, in which case we are done, or $Inf_{\varphi}(t,s_0)$. Since $s_0 < t+1$ there is some $s$ between $s_0$ and $t+1$ that satisfies $\varphi(t,s)$, so that $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ is true. □

The main step in the proof of the theorem is a proposition that states that every special formula $(\exists s)^{<t+1}_{\geq t}\varphi(t,s)$ is equivalent to a formula in $TLC$. We must first gather some more information about the $\varphi$-intervals.

**Lemma 3.5** *Given a functional formula $\varphi(t,s)$ there is an integer $q$ such that for every structure $M$ and every element $t_0$ in the structure, there are at most $q$ points $s$ to the right of $t_0$ which are right endpoints of a $\varphi$-interval $[t,s]$ that includes $t_0$. The integer $q$ is computable from $\varphi$ and it is simply the number $q$ of pairs in the decomposition of $\varphi$ in theorem 2.10.*

We call this integer the **nesting bound** of $\varphi$.

**Proof** By theorem 2.10 there is a finite set of pairs

$$\langle\alpha_1(x,z),\beta_1(z,y)\rangle,\cdots,\langle\alpha_q(x,z),\beta_q(z,y)\rangle$$

such that for every $t_0 \in (t,\ s)$

$$M \models \varphi(t,s) \qquad \text{iff} \qquad M \models \bigvee_{i=1}^{q}(\alpha_i(t,t_0) \wedge \beta_i(t_0,s))$$

Assume towards a contradiction that there are $q+1$ distinct points $s_1,\cdots,s_{q+1}$ to the right of $t_0$, corresponding to the left partners $t_1,\cdots,t_{q+1}$, on the left of $t_0$. Then for each pair $t_i,s_i$ there is a disjunct such that $M \models \alpha_{k_i}(t_i,t_0) \wedge \beta_{k_i}(t_0,s_i)$. Necessarily there are at least two elements $s_i,s_j$ for which $\beta_{k_i} = \beta_{k_j}$. But then $M \models \alpha_{k_i}(t_i,t_0) \wedge \beta_{k_i}(t_0,s_i)$ so that $\varphi(t_i,s_i)$ is true, and $M \models \alpha_{k_i}(t_i,t_0) \wedge$

$\beta_{k_i}(t_0, s_j)$ so that also $\varphi(t_i, s_j)$ is true. This contradicts the fact that $\varphi$ is functional. □

We say that $[a_1, b_1]$ is a **$\varphi$-interval of nesting at least** $i$ if there is a sequence $a_1 < \cdots < a_i < b_i < \cdots < b_1$ such that $\varphi(a_i, b_i)$ is true for all $j \leq i$. $[a_1, b_1]$ is a **$\varphi$-interval of nesting** $i$ if it is at least of nesting $i$ and it is not at least of nesting $i+1$.

Next we want to show that the properties which we investigate are expressible in plain temporal logic $TL(\mathbf{U}, \mathbf{S})$.

**Lemma 3.6** *Let $\varphi(t, s)$ be a functional formula.*

*There is a formula $R_\varphi(s)$ in first order monadic logic of order that says that $s$ is a right endpoint of a $\varphi$-interval. Moreover, for each $i$ there is a formula $R^i_\varphi(s)$ which says that $s$ is a right endpoint of a $\varphi$-interval of nesting $i$.*

*By Kamp's theorem there are also formulas of $TL(\mathbf{U}, \mathbf{S})$ which say the same things. We denote them also by $R_\varphi$ and $R^i_\varphi$.*

**Proof** Clearly

$$R_\varphi(s) \equiv \exists t \ (t \leq s \ \wedge \varphi(t, s))$$

Next we express the fact that $s$ is the right end point of a $\varphi$-interval of nesting at least $i$, denoting it by $R^{\geq i}_\varphi(s)$:

$$(\exists t, t_2, \cdots, t_i, s_2, \cdots, s_i) \left[(t < t_2 < \cdots < t_i \leq s_i < \cdots < s_2, < s) \wedge \varphi(t, s) \wedge \cdots \wedge \varphi(t_i, s_i)\right]$$

The required formula $R^i_\varphi(s)$ is:

$$R^i_\varphi(s) \ \equiv \ R^{\geq i}_\varphi(s) \ \wedge \ \neg R^{\geq (i+1)}_\varphi(s)$$

□

Next comes another auxiliary property:

**Lemma 3.7** *Let $\varphi(t, s)$ be a functional formula, with nesting bound $q$. For every structure $M$, point $t_0$ in $M$, and $i \leq q$, if some $s > t_0$ is a right endpoint of a $\varphi$-interval of nesting $i$, which contains $t_0$ then $s$ is one of the first $q$ points to the right of $t_0$ that satisfy $R^i_\varphi(s)$.*

**Proof** Suppose $(t, s)$ is a $\varphi$-interval of nesting $i$ containing a point $t_0$. We claim that any point $s'$, $t_0 < s' < s$, such that $R^i_\varphi(s')$ is the right endpoint of a $\varphi$-interval containing $t_0$. Indeed, let $(t', s')$ be a $\varphi$-interval of nesting $i$; then $t' < t_0$ since otherwise $(t', s')$ would be contained in $(t, s)$, and a $\varphi$-interval of nesting $i$ cannot be contained in another $\varphi$-interval of nesting $i$. This proves the claim, and it follows from Lemma 3.5 that $s$ must be one of the first $q$ points to the right of $t_0$ satisfying $R^i_\varphi(s)$. □

We are ready for the main proposition:

**Proposition 3.8** *If $\varphi(t, s)$ is a functional formula then the special formula $(\exists s)^{<t+1}_{\geq t} \varphi(t, s)$ is equivalent to a formula in $TLC$.*

**Proof** $(\exists s)^{<t+1}_{\geq t}\varphi(t,s)$ is equivalent to $\varphi(t,t) \vee (\exists s)^{<t+1}_{>t}\varphi(t,s)$. By the Kamp theorem, the first disjunct is equivalent to a $TL(\mathbf{U},\mathbf{S})$ formula. We are going to show that the second disjunct is equivalent to a $TLC$ formula. Therefore, $(\exists s)^{<t+1}_{\geq t}\varphi(t,s)$ is equivalent to a $TLC$ formula.

Suppose $\varphi(t,s)$ has a nesting bound $q$. For every $j$ we denote by $R^{i,j}_{\varphi}(t)$ the formula that says that the $j^{th}$ solution $s$ among those to the right of $t$ of the formula $R^i_{\varphi}(s)$ satisfies $\varphi(t,s)$. Thus $R^{i,j}_{\varphi}(t)$ is

$$(\exists s_1 \cdots \exists s_j) \quad \Big[\ (t < s_1 < \cdots < s_j)\ \wedge \varphi(t,s_j) \ \bigwedge\ \forall v[(t < v \leq s_j)\ \rightarrow\ (R^i_{\varphi}(v)\ \leftrightarrow\ \textstyle\bigvee_{i\leq j} v = s_i)]\Big]$$

From what we proved up to now, if $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ then this $s$ is also the solution to a formula $R^i_{\varphi}(s)$ for some $i \leq q$ and it must be one of the first $q$ solutions of $R^i_{\varphi}(s)$ to the right of $t$. Therefore $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ implies the following disjunction:

$$\bigvee_{i=1}^{q} [\bigvee_{j=1}^{q}\ ((\exists s_1,\cdots,s_j)^{<t+1}_{>t}\ s_1 < s_2 < \ldots < s_j \wedge \bigwedge_{k\leq j} R^i_{\varphi}(s_k)\ \wedge\ R^{i,j}_{\varphi}(t))]$$

On the other hand if this formula is satisfied then its witness $s_j$ is in particular smaller than $t+1$ and satisfies $\varphi(t,s_j)$. Therefore the two formulas are equivalent. We may now translate the formula into $TLC$:

$$\bigvee_{i=1}^{q} [\bigvee_{j=1}^{q}\ (C_j(\overline{R^i_{\varphi}})\ \wedge\ \overline{R^{i,j}_{\varphi}})]$$

where the overline is the natural translation of a simple monadic formula to temporal logic. □

Together with lemma 3.4 we proved that every simple metric formula $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ is equivalent to formula of $TLC$. We can now complete the proof of the main theorem:

**Proof of theorem 3.2**: Assume for contradiction that $\theta$ is a formula of *Q2MLO* with one free variable, with the smallest number of metric quantifications which is not equivalent to a formula of $TLC$. Since formulas without metric quantifiers are equivalent to $TL(\mathbf{S},\mathbf{U})$ formulas by Kamp's theorem we conclude that $\theta$ has at least one metric quantifier. We focus on some innermost such quantifier, and we assume that the quantifier is $(\exists s)^{<t+1}_{>t}$, and that it is the head of the subformula $\psi(t) \equiv (\exists s)^{<t+1}_{>t}\varphi(t,s)$, and $\varphi(t,s)$ has no metric quantifiers (the case $(\forall s)^{<t+1}_{>t}$ follows easily and the past quantifiers are treated similarly). Replacing this subformula by a new predicate symbol $X$ we obtain a formula $\theta'(X)$ such that $\theta$ is obtained from $\theta'$ by substituting $\psi(t)$ for $X$. It suffices therefore to prove that $\theta'(X)$ and $\psi(t)$ are equivalent to formulas $\overline{\theta'}(X)$ and $\overline{\psi}$ of $TLC$, so that $\theta$ is equivalent to the substitution of $\overline{\psi}$ for $X$ in $\overline{\theta'}(X)$.

$\theta'(X)$ is equivalent to a $TLC$ formula by the minimality of $\theta$ as a counter example. The formula $\psi(t) \equiv (\exists s)^{<t+1}_{>t}\varphi(t,s)$ is simple metric formula and it is equivalent to a formula of $TLC$ by the long discussion above. This concludes the proof.

# 4 Discussion and further results

We added to the temporal logic $TL(\mathbf{U},\mathbf{S})$ all the modalities $C_n(X)$ - "$X$ will be true at least at $n$ points in the next unit of time", and $\overleftarrow{C}_n(X)$ -"$X$ was true at least at $n$ points in the last unit of time". The resulting temporal logic is complete for a strong yet decidable monadic logic of order,*Q2MLO*. Some remarks are in order:

1. If we tried to prove directly that Pnueli's modalities can be expressed in $TLC$, i.e. in terms of counting, the proof would be similar to the general proof that we presented.

2. The proof does not use any special properties of first order monadic logic of order, except for the composition property. It applies to general logics that obey the composition rules. Specifically:

   **Notation:** Let $L$ be a logic. We denote by *Q2L* the minimal extension of $L$ defined as follows:

   (a) Every formula of $L$ is in *Q2L*.

   (b) *Q2L* is closed under Boolean connectors and first order quantifications

   (c) *Q2L* is closed under applications of the "metric quantifiers":

   If $\varphi(t_0,t)$ is a formula in *Q2L* with $t$ and $t_0$ its only free first-order variables and $m < n$ are integers then $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,t)$ is a formula of *Q2L* (in the free variable $t_0$).

   Assume that $L$ satisfies the composition theorem 2.10, with *Q2L* as described. Let $QLC$ be the sublogic of *Q2L* that avoids the metric quantifiers $(\exists s)^{<t+n}_{>t+m}$ in front of a formula with two free variables, and allows instead the quantifications which are definable in *Q2L*, $(\exists_n s)^{<t+1}_{>t}$ or $(\exists_n s)^{<t}_{>t-1}$ in front of a formula $\varphi$ that has only $s$ free, claiming that there are at least $n$ points in the unit interval ahead (or back) that satisfy $\varphi$.

   Then $QLC$ is expressively complete for *Q2L*. This is true whether we assume that the second order quantifiers in $L$ range over all subsets of the model, over finite subsets, over countable subsets, or over subsets with finite variability.

3. The proof does not use any property of the $+1$ function except for the fact that $t < t+1$. We do not have an example of non monotone function $h(t)$ with $t < h(t)$ for which it is interesting to replace $t+1$ in the proof

by $h(t)$, but it is worth remembering that not even monotonicity of the $+1$ function is used.

4. It seems that the expressive completeness result can be extended to the rational time line when the Stavi modalities are added (see [GHR94]). The adaptation is not entirely trivial since the proofs that we gave assumed that bounded sets have least upper bounds. We therefore leave it as a conjecture.

5. As with the pure temporal logic $TL(\mathbf{U}, \mathbf{S})$ there is a gap between the complexity (and succinctness) of the temporal logic and that of the corresponding predicate logic. Since *Q2MLO* contains the first-order monadic logic of order, the complexity of the satisfiability problem for *Q2MLO* is non-elementary. In [Rab08] it was shown that the satisfiability problem for the temporal logic $TL(\mathbf{U}, \mathbf{S}, \{C_n, \overleftarrow{C}_n\}_{n=1}^{\infty})$ is *PSPACE* complete under the unary coding of indices and it is *EXPSPACE* complete under the binary coding of indices.


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