THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

ON COUNTABLE CHAINS HAVING DECIDABLE MONADIC THEORY

ALEXIS BÈS AND ALEXANDER RABINOVICH

Abstract. Rationals and countable ordinals are important examples of structures with decidable monadic second-order theories. A chain is an expansion of a linear order by monadic predicates. We show that if the monadic second-order theory of a countable chain C is decidable then C has a non-trivial expansion with decidable monadic second-order theory.

§1. Introduction. The study of decidability of logical theories is a well-established research topic with numerous applications in Computer Science, in particular in the field of verification. Many techniques have been developed to build larger and larger classes of structures with a decidable theory. For an overview of recent related results in the framework of monadic second order (shortly: MSO) theories we refer e.g., to [4, 24]. It is interesting to explore the limit of specific decidability techniques, and also to prove general results about the frontier between decidability and undecidability.

In particular, Elgot and Rabin ask in [8] whether there exist maximal decidable structures, i.e., structures M with a decidable first-order (shortly: FO) theory and such that the FO theory of any expansion of M by a non-definable predicate is undecidable. This question is still open. Let us mention some partial results: Soprunov proved in [21] that every countable structure in which a regular ordering is interpretable is not maximal. A partial ordering $(B, <)$ is said to be regular if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a$, $b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable structure if we replace FO by *weak* MSO logic. In [2] it was shown that there exists a structure M with a decidable MSO theory and such that any expansion of M by a *constant symbol* has an undecidable FO theory. Paper [1] gives a sufficient condition in terms of the Gaifman graph of M which ensures that M is not maximal. The condition is the following: for every natural number r and every finite set X of elements of the domain $|M|$ of M there exists an element $x \in |M|$ such that the Gaifman distance between x and every element of X is greater than r.

In [3] we considered Elgot–Rabin's question for chains, i.e., linear orderings expanded with monadic predicates, in the framework of MSO theory. The class of chains is interesting with respect to the above results, since on the one hand

Received October 3, 2010.

no regular ordering seems to be interpretable in such structures (this intuition is supported by the fact that the full binary tree is not interpretable in a chain [18]), and on the other hand their associated Gaifman distance is trivial; thus, they do not satisfy the criterion given in [1]. We proved in [3] that for every chain $M = (A, <, \overline{P})$ such that $(A, <)$ contains a sub-interval of type ω or $-\omega$, M is not maximal with respect to MSO logic, i.e., there exists an expansion M' of M by a predicate which is not MSO definable in M, and such that the MSO theory of M' is recursive in the one of M.

In this paper we prove that this property holds for every infinite *countable* chain, namely that no infinite countable chain is maximal with respect to MSO logic. The proof relies on the composition method developed by Feferman–Vaught [9], Läuchli [15] and Shelah [20], which reduces the MSO theory of a sum of ordered structures to the one of its components.

The MSO logic of chains has a special interest as it provides prominent examples of decidable MSO theories, and also for the variety of approaches for proving decidability, such as Ehrenfeucht–Fraïssé games, automata, or composition methods (see e.g., [24]). Let us recall some important decidability results. In his seminal paper [5], Büchi proved that languages of ω-words recognizable by automata coincide with languages definable in the MSO theory of $(\omega, <)$, from which he deduced decidability of the theory. The result (and the automata method) was then extended to the MSO theory of any countable ordinal [6], to ω_1, and to any ordinal less than ω_2 [7]. Gurevich, Magidor and Shelah prove [12] that decidability of MSO theory of ω_2 is independent of ZFC. Let us mention results for linear orderings beyond ordinals. Using automata, Rabin [17] proved decidability of the MSO theory of the binary tree, from which he deduces decidability of the MSO theory of $\mathbb{Q}$, which in turn implies decidability of the MSO theory of the class of countable linear orderings. Shelah [20] improved model-theoretical techniques that allow him to reprove almost all known decidability results about MSO theories, as well as new decidability results for the case of linear orderings, and in particular dense orderings. He proved in particular that the MSO theory of $\mathbb{R}$ is undecidable. The frontier between decidable and undecidable cases was specified in later papers by Gurevich and Shelah [10, 13, 14]; we refer the reader to the survey [11].

§2. Preliminaries.

This section contains standard definitions, notations and some useful results.

2.1. Linear orderings and chains. We first recall useful definitions and results about linear orderings. A good reference on the subject is Rosenstein's book [19].

A *linear ordering* J is a total ordering. The order types of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are denoted by ω, ζ and η, respectively. Given a linear ordering J, we denote by $-J$ the *backwards* linear ordering obtained by reversing the ordering relation.

Given two elements j, k of a linear ordering J, we denote by $[j, k]$ (respectively (j, k)) the interval $[\min(j, k), \max(j, k)]$ (resp. $(\min(j, k), \max(j, k))$). An ordering is *dense* if it contains no pair of consecutive elements. An ordering I is *scattered* if there is no order-preserving mapping from η into I.

Given an ordering Y and a sub-ordering X of Y, we say that X is dense in Y if $[x, y] \cap X \neq \varnothing$ for every pair x, y of distinct elements of Y, and that X is nowhere dense in Y if for every open interval Z of Y, $X \cap Z$ is not dense in Z.

We say that X is co-dense in Y if $Y \setminus X$ is dense in Y.

In this paper we consider chains (or, labelled linear orderings), i.e., linear orderings $(A, <)$ equipped with a function $f\colon A \to T$ where T is a finite (nonempty) set.

Given a dense ordering I, a finite set T, and a coloring $C\colon I \to T$, we say that an interval $J \subseteq I$ is C-uniform if for every $t \in T$ the set $J \cap C^{-1}(t)$ is either empty or dense in J. We shall use the following result (see [15]).

PROPOSITION 2.1. *Let I be a dense ordering. For every finite set T and every coloring $C\colon I \to T$, I contains an infinite C-uniform interval.*

2.2. Logic. Let us briefly recall useful elements of monadic second-order logic, and settle some notations. For more details about MSO logic see e.g., [11, 23]. Monadic second-order logic is an extension of first-order logic that allows to quantify over elements as well as subsets of the domain of the structure. Given a signature L, one can define the set of (MSO) formulas over L as well-formed formulas that can use first-order variable symbols $x, y, \dots$ interpreted as elements of the domain of the structure, monadic second-order variable symbols $X, Y, \dots$ interpreted as subsets of the domain, symbols from L, and a new binary predicate $x \in X$ interpreted as "x belongs to X". A sentence is a formula without free variable. As usual, we often identify logical symbols with their interpretation. Given a signature L and an L-structure M with domain D, we say that a relation $R \subseteq D^m \times (2^D)^n$ is (MSO) definable in M if and only if there exists a formula $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$ over L which is true in M if and only if $(x_1, \dots, x_m, X_1, \dots, X_n)$ is interpreted by an $(m+n)$-tuple of R. Given a structure M we denote by $\mathrm{MSO}(M)$ (respectively $\mathrm{FO}(M)$) the monadic second-order (respectively first-order) theory of M.

We say that M is *maximal* if $\mathrm{MSO}(M)$ is decidable and $\mathrm{MSO}(M')$ is undecidable for every expansion M' of M by a predicate which is not definable in M.

We can identify labelled linear orderings with structures of the form $M = (A, <, P_1, \dots, P_n)$ where $<$ is a binary relation interpreted as a linear ordering over A, and the P_i's denote unary predicates. We use the notation $\overline{P}$ as a shortcut for the n-tuple $(P_1, \dots, P_n)$.

Let Σ and Σ' be relational signatures, M a Σ-structure with domain A and M' a Σ'-structure with domain A'. We say that M is (MSO) interpretable in M' if there exist a subset D of A' and a surjective map $\mathscr{I}\colon D \to A$ such that:

- D is MSO definable in M';
- The equivalence relation $EQ_{\mathscr{I}} = \{(x, y) \in A' \colon \mathscr{I}(x) = \mathscr{I}(y)\}$ is MSO definable in M';
- For every m-ary symbol R of Σ, there exists a MSO Σ'-formula φ_R such that

$$M \models R(\mathscr{I}(a_1), \dots, \mathscr{I}(a_m)) \Leftrightarrow M' \models \varphi_R(a_1, \dots, a_m)$$

  for all $a_1, \dots, a_m \in D$.

The following property of interpretations is well-known.

LEMMA 2.2. *If M is interpretable in M' then* $\mathrm{MSO}(M)$ *is recursive in* $\mathrm{MSO}(M')$.

Let us recall the following result.

THEOREM 2.3 (Rabin [17]). $\mathrm{MSO}(\eta, <)$ *is decidable.*

We shall use the following easy corollary of Theorem 2.3.

COROLLARY 2.4. *Let $M = (\eta, <, P_1, \dots, P_n)$ be such that $(P_1, \dots, P_n)$ is a partition and every P_i is non-empty and dense in η. Then* MSO(M) *is decidable.*

PROOF. We prove that MSO(M) is recursive in MSO$(\eta, <)$, and use Theorem 2.3. For every n, all structures $(\eta, <, P_1, \dots, P_n)$ such that $(P_1, \dots, P_n)$ is a partition and every P_i is non-empty and dense in η, are isomorphic. Moreover there exists an MSO-formula $U(X_1, \dots, X_n)$ which expresses that $(X_1, \dots, X_n)$ is a partition and that every X_i is non-empty and dense. Hence for every sentence φ, we obtain that $M \models \varphi$ iff

$$(\eta, <) \models \exists X_1 \dots \exists X_n (U(X_1, \dots, X_n) \wedge \varphi^*))$$

where φ^* is obtained from φ by replacing every atomic formula of the form $P_i(x)$ by $x \in X_i$. ⊣

2.3. Elements of composition method. In this paper we rely heavily on composition methods, which allow to compute the theory of a sum of structures from the ones of its summands. For an overview of the subject see [4, 22, 16]. In this section we recall useful definitions and results. The quantifier depth of a formula φ is denoted by $qd(\varphi)$. Let $n \in \mathbb{N}$, Δ any finite signature that contains only relational symbols. and M_1, M_2 be Δ-structures. We say that M_1 and M_2 are *n-equivalent*, denoted $M_1 \equiv^n M_2$, if for every sentence φ of quantifier depth at most n, $M_1 \models \varphi$ iff $M_2 \models \varphi$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and Δ, the set of sentences of quantifier depth $\leq n$ is infinite. However, it contains only finitely many semantically distinct sentences, so there are only finitely many $\equiv^n$-classes of Δ-structures. In fact, we can compute representatives for these classes.

LEMMA 2.5 (Hintikka Lemma). *For each $n \in \mathbb{N}$ and a finite signature Δ that contains only relational symbols, we can compute a* finite *set $H_n(\Delta)$ of Δ-sentences of quantifier depth at most n such that*:

- *If $\tau_1, \tau_2 \in H_n(\Delta)$ and $\tau_1 \neq \tau_2$, then $\tau_1 \wedge \tau_2$ is unsatisfiable.*
- *If $\tau \in H_n(\Delta)$ and $qd(\varphi) \leq n$, then $\tau \to \varphi$ or $\tau \to \neg\varphi$. Furthermore, there is an algorithm that, given such τ and φ, decides which of these two possibilities holds.*
- *For every Δ-structure M there is a* unique *$\tau \in H_n(\Delta)$ such that $M \models \tau$.*

Elements of $H_n(\Delta)$ are called (n, Δ)-Hintikka sentences.

Given a Δ-structure M we denote by $T^n(M)$ the unique element of $H_n(\Delta)$ satisfied in M and call it the *n-type* of M. Thus, $T^n(M)$ determines (effectively) which sentences of quantifier-depth $\leq n$ are satisfied in M.

As a simple consequence, note that the MSO theory of a structure M is decidable if and only if the function $k \mapsto T^k(M)$ is recursive.

The sum of chains corresponds to concatenation. Let us recall a general definition.

DEFINITION 2.6 (sum of chains). Consider an index structure $Ind = (I, <^I)$ where $<^I$ is a linear ordering. Consider a signature $\Delta = \{<, P_1, \dots, P_l\}$, where P_i are unary predicate names, and a family $(M_i)_{i \in I}$ of Δ-structures $M_i = (A_i; <^i, {P_1}^i, \dots, P_l^i)$ with disjoint domains and such that the interpretation $<^i$ of $<$ in each M_i is a linear ordering. We define the *ordered sum* of the family $(M_i)_{i \in I}$ as the Δ-structure $M = (A; <^M, {P_1}^M, \dots, P_l^M)$ where

- A equals the union of the A_i's
- $x <^M y$ holds if and only if ($x \in A_i$ and $y \in A_j$ for some $i <^I j$), or ($x, y \in A_i$ and $x <^i y$)
- for every $x \in A$ and every $k \in \{1, \dots, l\}$, $P_k^M(x)$ holds if and only if $M_j \models P_k^j(x)$ where j is such that $x \in A_j$.

If the domains of the M_i are not disjoint, replace them with isomorphic chains that have disjoint domains, and proceed as before.

We shall use the notation $M = \sum_{i \in I} M_i$ for the ordered sum of the family $(M_i)_{i \in I}$.

If $I = \{1, 2\}$ has two elements, we denote $\sum_{i \in I} M_i$ by $M_1 + M_2$.

We shall use Shelah's composition method [20, Theorem 2.4] (see also [11, 22]) which allows to reduce the MSO theory of a sum of chains to the MSO theories of the summands and the MSO theory of the index structure.

THEOREM 2.7 (Composition Theorem [20]). *There exists a recursive function f and an algorithm which, given $k, l \in \mathbb{N}$, computes the k-type of any sum $M = \sum_{i \in I} M_i$ of chains over a signature $\{<, P_1, \dots, P_l\}$ from the $f(k, l)$-type of the structure*

$$(I, <^I, Q_1, \dots, Q_p)$$

where

$$Q_j = \{i \in I \colon T^k(M_i) = \tau_j\} \quad j = 1, \dots, p$$

and $\tau_1, \dots, \tau_p$ is the list of all (k, Δ)-Hintikka sentences with $\Delta = \{<, P_1, \dots, P_l\}$.

The two following results ([20, Sections 5 and 6], see also [25, Theorem 5.6 p. 41]) specifies Theorem 2.7 in case $I = \eta$ and all sets Q_i are either empty or dense in η.

THEOREM 2.8 (Shuffle). *Let $k, l \in \mathbb{N}$, and S be a nonempty set of k-types over the signature $\Delta = \{<, P_1, \dots, P_l\}$. For every sum $M = \sum_{i \in \eta} M_i$ of chains over Δ such that $S = \{T^k(M_i) \colon i \in \eta\}$, and $\{i \in \eta \colon T^k(M_i) = \tau\}$ is dense in η for every $\tau \in S$, the k-type of M is completely determined by S, k and l. Moreover it can be computed from S, k and l. This k-type is called* shuffle of *S and is denoted by shuffle(S).*

2.4. Decomposition of a chain. Let M be a chain and let $\sim$ be an equivalence relation on the domain of M. If the $\sim$-equivalence classes are intervals in M we say that $\sim$ is a *convex* equivalence relation. In this case the set of $\sim$-equivalence classes can be naturally ordered by $i_1 \leq i_2$ iff $\exists x_1 \in i_1 \exists x_2 \in i_2 (x_1 \leq x_2)$. We denote by $M/_\sim$ the linear order of $\sim$-equivalence classes. The mapping that assigns to every $x \in M$ its $\sim$-equivalence class is said to be *canonical.*

Let $\sim$ be a convex equivalence relation on M. Then $M = \sum_{i \in M/_\sim} M_i$, where M_i is the subchain of M over the equivalence class i.

LEMMA 2.9. *If $\sim$ is a convex equivalence relation which is definable in M, then*

1. *$M/_\sim$ is interpretable in M.*
2. *Let $\varphi_1, \dots \varphi_k$ be sentences in the signature of M. Let a chain C be the expansion of $M/_\sim$ by unary predicates $Q_{\varphi_1}, \dots, Q_{\varphi_k}$ defined as*

$$Q_{\varphi_l} = \{i \in M/_\sim \mid M_i \models \varphi_l\}.$$

Then C is interpretable in M.

§3. Non-maximality for MSO theories of countable chains.

3.1. Main result. An expansion of M by a predicate R is non-trivial if R is not MSO-definable in M.

The next theorem is our main result.

THEOREM 3.1. *Let $M = (A, <, \overline{P})$ be an infinite countable chain. There exists a non-trivial expansion M' of M by a monadic predicate such that* $\mathrm{MSO}(M')$ *is recursive in* $\mathrm{MSO}(M)$. *In particular if* $\mathrm{MSO}(M)$ *is decidable, then* $\mathrm{MSO}(M')$ *is decidable.*

In this section we prove Theorem 3.1. We shall use the following result from [3].

LEMMA 3.2. *Let $M = (A, <, \overline{P})$ be an infinite chain which contains an interval of order type ω or $-\omega$. There exists a non-trivial expansion M' of M such that* $\mathrm{MSO}(M')$ *is recursive in* $\mathrm{MSO}(M)$.

In the rest of this section we prove Theorem 3.1.

Let $\overline{P} = (P_1, \dots, P_t)$. We can assume w.l.o.g. that $\overline{P}$ is a partition of A. The structure M' will be defined as the expansion of M with some unary predicate R.

Consider the equivalence relation $\approx$ defined on A which holds between x and y if either $[x, y]$ is finite, or $[x, y]$ is contained in an open dense interval which is C-uniform with respect to the coloring $C\colon A \to \{1, \dots, t\}$ which maps every $x \in A$ to the unique i such that $x \in P_i$. Observe that the relation $x \approx y$ is MSO-definable in M. Each $\approx$-equivalence class has one of the following forms:

1. orderings of type $-\omega$, or ω, or ζ;
2. an interval of order type η which is C-uniform;
3. finite orderings.

We denote by J the linear order $M/_{\approx}$ of the $\approx$-equivalence classes. We can write $M = \sum_{j \in J} M_j$ (respectively $M' = \sum_{j \in J} M'_j$), where for every $j \in J$ the domain of M_j (resp. M'_j) corresponds to an $\approx$-equivalence class.

Hence, at least one of the following cases holds:

1. At least one $\approx$-equivalence class has order-type $-\omega$, or ω, or ζ;
2. at least one $\approx$-equivalence class has order-type η;
3. all $\approx$-equivalence classes are finite.

We prove Theorem 3.1 for each of these cases separately.

If there exists at least one $\approx$-equivalence class of order-type $-\omega$, or ω, or ζ, then the result follows from Lemma 3.2.

The case when at least one $\approx$-equivalence class has order-type η is considered in the next subsection. The case when all $\approx$-equivalence classes are finite is considered in subsection 3.3.

3.2. Second case: there exists at least one $\approx$-equivalence class of order type η. In this case we can expand M with any predicate R which satisfies the following conditions:

1. If j is an $\approx$-equivalence class of type (1) or (3) then $R \cap j = \varnothing$.
2. If j is an $\approx$-equivalence class of type (2) and $Y_j = \{l \mid P_l \cap j \neq \varnothing\}$, then R is dense and co-dense in $j \cap P_l$ for every $l \in Y_j$.

LEMMA 3.3. *The set R is not definable in M.*

Proof. Assume that a unary predicate H is definable in a chain M by an MSO formula $\varphi(x)$. Let g be an automorphism on M. Then H should be invariant under g, i.e., g maps H onto H.

Let j be an $\approx$-equivalence class of type (2). It is order-isomorphic to η. Let M_j be the substructure of M over j. For every $l \in Y_j$, P_l is dense and co-dense in j. Then, by (2), there are $a_1, a_2 \in j$ such that $a_1 \in P_l \cap R$ and $a_2 \in P_l \setminus R$ for some $l \in Y_j$, and there exists an automorphism f of M_j, which maps a_1 to a_2. Hence, R is not invariant under f. We can extend f to an automorphism g of M. Since R is not invariant under g, we derive that R is not MSO-definable in M. ⊣

The next definition introduces notations which will be used throughout the paper.

Definition 3.4 (Chains N_k and N'_k). Let M be a chain in a signature Δ and let M' be an expansion of M by a predicate R. For $k > 0$ we define chains N_k and N'_k as follows. Let $(J, <)$ be the chain $M/_\approx$ of $\approx$-equivalence classes.

1. N_k is the expansion $(J, <)$ by predicates $\{Type_{k,\tau} \mid \tau \in H_k(\Delta)\}$ defined as: $Type_{k,\tau}(j)$ holds iff $T^k(M_j) = \tau$.
2. N'_k is the expansion $(J, <)$ by predicates $\{Type'_{k,\tau} \mid \tau \in H_k(\Delta \cup \{R\})\}$ defined as: $Type'_{k,\tau}(j)$ holds iff $T^k(M'_j) = \tau$.

Note that N_k and N'_k are chains over the same domain, however they have different signature. The following lemma is a consequence of Lemma 2.9.

Lemma 3.5. 1. *N_k is interpretable in M.*
2. *N'_k is interpretable in M'.*
3. *N_k is interpretable in N_m for every $m \geq k$.*

Lemma 3.6. MSO(M') *is recursive in* MSO(M).

Proof. We show how to reduce $T^n(M')$ to MSO(M) for every $n \geq 3$. Note that $T^0(M')$ is always empty (since there is no sentence with quantifier depth 0 in the signature of M'), and moreover $T^1(M')$ and $T^2(M')$ clearly reduce to $T^3(M')$. The main reduction steps can be represented as follows:

$$T^n(M') \to \text{MSO}(N'_n) \to \text{MSO}(N_n) \to \text{MSO}(M).$$

Let $n \geq 3$. By Theorem 2.7, $T^n(M')$ is recursive in MSO(N'_n).

By Lemma 3.5 there is an interpretation of N_n in M, therefore MSO(N_n) is recursive in MSO(M).

It remains to show that MSO(N'_n) is recursive in MSO(N_n).

Let us prove that for every $j \in J$, $T^n(M'_j)$ can be computed from $T^n(M_j)$. First of all, using $T^3(M_j)$ we can check whether the $\approx$-class j has type (2). Indeed, only classes of type (2) are dense, thus

$$T^3(M_j) \to \forall x \forall y (x < y \to \exists z (x < z \wedge z < y))$$

iff j has type (2).

If j is not of type (2), then by definition of R we have $R \cap j = \varnothing$. In this case for every sentence φ we have $M'_j \models \varphi$ iff $M_j \models \varphi^*$ where φ^* is obtained from φ by replacing every atomic formula of the form $R(x)$ by $\neg(x = x)$. This shows that in this case $T^n(M'_j)$ can be computed from $T^n(M_j)$.

Assume now that j has type (2). Let $Y_j = \{i : P_i \cap j \neq \varnothing\}$. The set Y_j is computable from $T^1(M_j)$. Let us denote by $i_1, \dots, i_t$ the (distinct) elements of Y_j. For $u = 1, \dots, t$, let $P_{i_u,1} = P_{i_u} \cap R$ and $P_{i_u,2} = P_{i_u} \setminus R$. It follows from our assumptions that all sets $P_{i_u,1}$ and $P_{i_u,2}$ are non-empty and dense in the domain of M_j (which we identify with η). By Corollary 2.4 it follows that the MSO theory of the structure $S = (\eta, <, P_{i_1,1}, P_{i_2,2}, \dots, P_{i_t,1}, P_{i_t,2})$ is decidable. We have $R = \cup_{1 \le u \le t} P_{i_u,1}$, and $P_{i_u} = P_{i_u,1} \cup P_{i_u,1}$, thus R and all predicates $P_{i,u}$ are MSO-definable in S. It follows that $\mathrm{MSO}(M'_j)$ is recursive in $\mathrm{MSO}(S)$.

We proved that for every $j \in J$, $T^n(M'_j)$ is computable from $T^n(M_j)$. This implies that every predicate $\mathit{Type}'_{k,\tau}$ in the signature of N'_n is equivalent to a boolean combination of predicates $\mathit{Type}_{k,\tau}$ in the signature of N_n, and thus is definable in N_n. It follows that N'_n is interpretable in N_n, and Lemma 2.2 yields that $\mathrm{MSO}(N'_n)$ is recursive in $\mathrm{MSO}(N_n)$. ⊣

This completes the proof for the second case.

3.3. Third case: all $\approx$-equivalence classes are finite. The construction in this case shares some ideas with the previous one but is more involved.

Since every $\approx$-equivalence class is finite, there are no consecutive $\approx$-equivalence classes. Therefore, the ordering J of $\approx$-equivalence classes is infinite, countable and dense (i.e., it is isomorphic with η, $1 + \eta$, $\eta + 1$ or $1 + \eta + 1$).

We shall expand M with a unary predicate R which will be defined "at the level of J", i.e., for every j we will have $(j \cap R) \in \{\varnothing, j\}$. Thus we actually define a set $R' \subseteq J$, and then define R by: $j \cap R = j$ if $j \in R'$, and $j \cap R = \varnothing$ otherwise.

For every $n > 0$ let C_n denote the coloring which maps every $j \in J$ to $T^n(M_j)$. Consider the equivalence relation $j \sim_n k$ which holds between elements $j, k \in J$ iff ($j = k$, or there exists a C_n-uniform open interval of J which contains both j and k). Each $\sim_n$-equivalence class is either a singleton, which we call an *n-point*, or a (maximal) C_n-uniform open interval, which we call an *n-interval*. If I is an n-interval and $S = \{\tau \mid \text{there is } j \in I \text{ such that } T^n(M_j) = \tau\}$, then I is said to be an *S-interval*. Note that if I is an S-interval and $\tau \in S$, then $\{j \mid T^n(M_j) = \tau\}$ is dense in I.

The main idea is to define R' in such a way that the following property holds:

(*) For every $n > 0$, every n-interval I of J,
and every n-type $\tau \in \{T^n(M_i) \mid i \in I\}$,
the set R' is both dense and co-dense in $I \cap \{i \mid T^n(M_i) = \tau\}$.

This property will ensure that R' is not definable in M (see Proposition 3.12), and on the other hand, will allow to reduce the computation of the n-type of the expansion of M by R to $\mathrm{MSO}(M)$ (see Proposition 3.13).

For every $n > 0$ let Π_n (respectively, I_n) denote the set of n-points (respectively n-intervals), and let $\Pi = \bigcup_n \Pi_n$. The definition of R' proceeds in two main stages: we first define the restriction of R' to $J \setminus \Pi$, and then the restriction of R' to Π (by defining it on every Π_n, by induction over n).

The following is easy:

Lemma 3.7 (properties of Π_n). 1. *Π_n is* MSO *definable in N_m for $m \ge n$.*
2. *Π_n is nowhere dense in J.*

Proof. (1) is immediate. (2) Assume for a contradiction that Π_n is dense in some open interval I of J. Then by Proposition 2.1, the interval I contains some

C_n-uniform open subinterval I', and I' is simultaneously contained in some n-interval, and contains n-points, which is impossible. ⊣

First stage: definition of R' on $J \setminus \Pi$:
Let $J' = J \setminus \Pi$. If J' is empty then we are done. Otherwise, let $\approx_1$ be an equivalence relation on J defined as follows: $x \approx_1 y$ if $x = y$ or there is an open interval $I \subseteq J$ of order type η such that $x, y \in I$, M_x and M_y are isomorphic, and the set $\{z \in I \setminus \Pi \mid M_z \text{ and } M_x \text{ are isomorphic}\}$ is dense in I. Note that an $\approx_1$-equivalence class is either a singleton or of order type η.

Now we define R' on J' as any set that contains all singleton $\approx_1$-equivalence classes and is dense and co-dense in every $\approx_1$-equivalence class of order type η.

The following lemma is crucial in order to prove $(*)$.

Lemma 3.8. *Let $n > 0$, I be an open sub-interval of an n-interval, τ be an n-type which appears (densely and co-densely) in I, and*

$$Y_\tau = I \cap \{j \colon T^n(M_j) = \tau\}.$$

If $Y_\tau \cap \Pi = \varnothing$, then every element x of Y_τ belongs to an $\approx_1$-equivalence class C of order type η, and Y_τ has a non-empty intersection with both R' and its complement.

Proof. Let $x \in Y_\tau$. The structure M_x is finite. Therefore, there is a sentence φ such that a chain satisfies φ iff it is isomorphic to M_x. Let $m = qd(\varphi)$. Since $x \notin \Pi_m$, it follows that x belongs to an m-interval I_m. Hence, the set $E_x = \{z \mid M_z \text{ and } M_x \text{ are isomorphic}\}$ is dense in $I_m \cap I$. Now every element z of E_x satisfies $T^n(M_z) = \tau$, thus $E_x \cap I \subseteq Y_\tau$. By our assumption $Y_\tau \cap \Pi = \varnothing$, from which it follows that all elements of $E_x \cap I \cap I_m$ (which is an infinite set) are $\approx_1$-equivalent to x. Thus the $\approx_1$-equivalence class of x is of order type η, hence $E_x \cap I \cap I_m$ has a non-empty intersection with both R' and its complement. ⊣

Second stage: definition of R' on Π:
We define by induction on n the set $R'_n \subseteq \Pi_n \setminus \Pi_{n-1}$, and define then the restriction of R' to Π as $\cup_{n \in \mathbb{N}} R'_n$.

Let us first explain the definition informally. We want that eventually R' satisfies $(*)$. Let us start with a simple example. Consider the case $n = 1$, and the partition of J into 1-intervals and 1-points. Consider a 1-interval I. It does not contain any 1-point (by definition), but it can contain m-points for some $m > 1$. Thus if we want that $(*)$ holds for $n = 1$ and I, we have to ensure that the definition of R' for m-points is compatible with $(*)$. If the set $I \cap \Pi$ is finite, or even nowhere dense in I, then the definition of R' on $I \setminus \Pi$ given during the first stage suffices to ensure that $(*)$ holds for $n = 1$. Thus we could simply choose to put all elements of $I \cap \Pi$ in R' (or all in the complement of R'). However it can happen that *all* elements of I belong to Π. Thus we need some convenient strategy for defining R' on $I \cap \Pi$.

Let us consider the following example.

Example 3.1. Let $(A_j)_{j>0}$ be a family of disjoint subsets of η. For every $i > 0$ let $A_{\leq i} = \cup_{j=1}^{i} A_j$. Assume that the two following properties hold:

1. for every $i > 0$, $A_{\leq i}$ is order-isomorphic to a subset of the integers.
2. $\bigcup_{i>0} A_i$ is dense in η.

We define the sets $R_i \subseteq A_i$ as follows:

- $R_1 = \varnothing$
- for every $i > 0$, $x \in R_{i+1}$ iff $x \in A_{i+1}$ and there are $y, z \in A_{\leq i}$ such that
  1. $y < x < z$ and $(y, z) \cap A_{\leq i} = \varnothing$ (note that this implies that y and z are unique) and
  2. y and z belong to the complement of $\cup_{j=1}^{i} R_j$.

It is easy to see that R_i is MSO-definable in $(\eta, <)$ with parameters $A_1, \ldots, A_i$. Let us show that the set $R = \cup_{i>0} R_i$ is dense and co-dense in the set $A = \cup_{i>0} A_i$.

Take any open interval $I \subseteq \eta$. We have to show that there is a point in $I \cap A \cap R$ and a point in $(I \cap A) \setminus R$.

Toward a contradiction assume that $I \cap A \cap R = \varnothing$. Since A is dense in η there is i such that $I \cap A_{\leq i}$ contains at least two points. These points do not belong to R. Let us consider two consecutive points $y < z$ in $I \cap A_{\leq i}$. Since $I \cap A \cap R = \varnothing$ we obtain that $y, z \notin \cup_{j=1}^{i} R_j$. Since A is dense in (y, z) there is a minimal $m > i$ such that $A_m \cap (y, z) \neq \varnothing$. Then according to the definition of R_m all points of $A_m \cap (y, z)$ should be in R_m and hence in R. This contradicts the assumption that $I \cap A \cap R = \varnothing$.

Similar arguments show that $(I \cap A) \setminus R \neq \varnothing$. Hence R is dense and co-dense in A.

Our definition of R'_i refines the definition of R_i in the above example. First obstacle we have to overcome is to generalize the definition given for the family $(A_i)_{i>0}$ to the family $(\Pi_i)_{i>0}$. Note that a set Π_i is not necessarily order isomorphic to a subset of integers, though Π_i is nowhere dense. The second obstacle, in order to prove $(*)$, is that even in the case when $J = \Pi = \eta$, we have to ensure that R' is dense in $\eta \cap Y_\tau$ and the construction in the example only ensures that R is dense in η.

For every $n > 1$ let $A_n = \Pi_n \setminus \Pi_{n-1}$. We say that y is an (m, n)-left bound for x (and denote it as $BL_m^n(y, x)$) if the following conditions hold:

1. $y < x$
2. $x \in A_n$
3. $y \in \Pi_{n-1}$
4. $[y, x]$ is a subinterval of some m-interval and $T^m(M_x) = T^m(M_y)$

Note that by (4) if $BL_m^n(y, x)$ then $m < n$. The predicate $BR_m^n(y, x)$ for the relation "y is an (m, n)-right bound for x" is defined similarly.

Define $lrank_m^n(x)$ and $rrank_m^n(x)$ as

$$lrank_m^n(x) := \exists y BL_m^n(y, x) \wedge \neg \exists y BL_{m+1}^n(y, x).$$
$$rrank_m^n(x) := \exists y BR_m^n(y, x) \wedge \neg \exists y BR_{m+1}^n(y, x).$$

If $lrank_m^n(x)$ (respectively, $rrank_m^n(x)$) we say that the left rank (respectively, right rank) of x is (m, n).

We are going to define $R'_n \subseteq A_n$ by induction on n.

We say that $R'_{<n}$ holds at the left bound for x if $lrank_m^n(x)$ and

- either $\{y \mid BL_m^n(y, x)\}$ has a maximal element z and $z \in \cup_{j=1}^{n-1} R'_j$
- or $\{y \mid BL_m^n(y, x)\} \cap \cup_{j=1}^{n-1} R'_j$ is co-final in $\{y \mid BL_m^n(y, x)\}$.

One defines similarly "$R'_{<n}$ holds at the right bound for x".

We define the sets $R'_i \subseteq A_i$ as follows

- $R'_1 = \varnothing$.
- For every $i > 0$ and $x \in A_{i+1}$, $x \notin R'_{i+1}$ iff ($lrank^{i+1}_{m_1}(x)$ and $rrank^{i+1}_{m_2}(x)$ hold for some m_1, m_2, and $R'_{<i+1}$ holds at the left bound for x, and $R'_{<i+1}$ holds at the right bound for x).

Recall that the structure N_k was defined in Definition 3.4. The following lemma is immediate.

LEMMA 3.9. *For every $m \geq n \geq 1$ the relation R'_n is* MSO *definable in N_m.*

The following lemma describes a property of R' which is an instance of (*) and is central for our proof of (*).

LEMMA 3.10. *Let $n > 0$ and let I be an open subinterval of an n-interval. Let τ be an n-type such that the set $Y_\tau = I \cap \{x : T^n(M_x) = \tau\}$ is dense in I. Assume that Π is dense in Y_τ. Then, R' is both dense and co-dense in Y_τ.*

PROOF. We are going to prove that Y_τ contains a point in R' and a point outside R'. Since I is an arbitrary non-empty open interval this implies the conclusion of the lemma.

Since Π is dense in Y_τ, the set $\Pi \cap Y_\tau$ contains at least two points $a < b$. Let $a \in A_{k_1} \cap Y_\tau$ and $b \in A_{k_2} \cap Y_\tau$. Note that $k_1, k_2 > n$.

Toward a contradiction assume

$$R' \cap Y_\tau \cap (a, b) = \varnothing. \tag{1}$$

We shall prove the following:

$$\begin{array}{c}\text{for every } s > \max(k_1, k_2) \text{ and every } x \in A_s \cap Y_\tau \cap (a, b),\\ \text{there exists } l \leq k_1 \text{ such that } lrank^s_l(x) \text{ holds.}\end{array} \tag{2}$$

Let s, x be as in (2). Both elements a and x belong to I, thus $[a, x]$ is a subinterval of an n-interval. Moreover we have $T^n(M_a) = T^n(M_x)$ since $a, x \in Y_\tau$. Finally we have $a \in A_{k_1}$ with $k_1 < s$, thus $a \in \Pi_{s-1}$. It follows that $BL^s_n(a, x)$ holds. Therefore, $lrank^s_l(x)$ holds for some $l \geq n$. Since $a \in \Pi_{k_1}$, by condition 4 in the definition of BL^s_l we obtain that $\neg BL^s_{k_1}(a, x)$. Therefore,

$$\text{If } l > k_1 \text{ and } y \text{ satisfies } BL^s_l(y, x) \text{ then } y > a. \tag{3}$$

By (1) we have $x \notin R'_s$, therefore by the definition of R'_s, we obtain that $R'_{<s}$ holds at the left bound for x. If $l > k_1$ then by (3) and (1), it follows that $R'_{<s}$ does not hold at the left bound for x. Hence, a contradiction. Therefore, $l \leq k_1$, which yields (2).

Recall that Π is dense in Y_τ. It follows from (2) that there exists $l_1 \leq k_1$ and a non-empty open interval $V \subseteq (a, b)$ such that $\{x \in \Pi \mid lrank^s_{l_1}(x) \wedge s > k_1\}$ is dense in $V \cap Y_\tau$.

Since for every i the set Π_i is nowhere dense and $A_{i+1} = \Pi_{i+1} \setminus \Pi_i$, we obtain that for every r, there exist integers $s_1 < s_2 < \cdots < s_r$ and elements $x_1 < x_2 < \cdots < x_r$ of $V \cap Y_\tau$ such that $lrank^{s_i}_{l_1}(x_i)$ holds for every i.

Let u be the number of $(l_1, \{<, \overline{P}\})$-Hintikka sentences, and let $r > 2u$. We obtain that there is an l_1-type τ' and $x_i < x_p < x_j$ such that

$$\tau' = T^{l_1}(M_{x_i}) = T^{l_1}(M_{x_j}) = T^{l_1}(M_{x_p}).$$

First observe

(4) $$\{y \mid y > x_i \wedge BL^{s_j}_{l_1}(y, x_j)\} \neq \varnothing$$

Indeed, on the one hand if $BL^{s_j}_{l_1}(x_i, x_j)$, then (x_i, x_j) is a sub-interval of an l_1-interval, and therefore $BL^{s_j}_{l_1}(x_p, x_j)$. Since $x_i < x_p$ it follows that

$$x_p \in \{y \mid y > x_i \wedge BL^{s_j}_{l_1}(y, x_j)\},$$

and therefore (4) holds.

On the other hand, if $\neg BL^{s_j}_{l_1}(x_i, x_j)$ then (x_i, x_j) is not a sub-interval of any l_1-interval, and in this case $\{y \mid y > x_i \wedge BL^{s_j}_{l_1}(y, x_j)\} = \{y \mid BL^{s_j}_{l_1}(y, x_j)\} \neq \varnothing$.

Next, observe that, by (1), no element of $(a, x_j) \cap Y_\tau$ belongs to R', and hence no element of $(x_i, x_j) \cap Y_\tau$ belongs to R'.

Recall that for every $s > \max(k_1, k_2)$ and every $x \in A_s \cap Y_\tau \cap (a, b)$ we proved that $lrank^s_l(x)$ holds for some $l \geq n$. Therefore, $l_1 \in [n, k_1)$ and $\tau' \to \tau$. In addition, (x_i, x_j) is an sub-interval of an n-interval, therefore $(x_i, x_j) \cap Y_{\tau'} \subseteq (x_i, x_j) \cap Y_\tau$. Hence, no element of $(x_i, x_j) \cap Y_{\tau'}$ belongs to R', and

(5) $$\{y \mid y > x_i \wedge BL^{s_j}_{l_1}(y, x_j)\} \cap R' = \varnothing.$$

Finally by (4) and (5), $R'_{<s_j}$ does not hold at the left bound for x_j, and by the definition of R'_{s_j}, we obtain that $x_j \in R'_{s_j}$. This contradicts (1).

We have proved that Y_τ contains a point in R'. The proof that Y_τ contains a point outside R' is similar. ⊣

LEMMA 3.11. *R' satisfies* $(*)$.

PROOF. Let $n > 0$ and let I be some n-interval. Then I is an S-interval for some set $S = \{\tau_1, \dots, \tau_p\}$ of n-types. We have to prove that for every $\tau \in S$ the set R' is both dense and co-dense in the set $Y_\tau = I \cap \{x : T^n(M_x) = \tau\}$. Let us fix τ, and let us consider an open interval Z of I. We shall prove that $Z \cap Y_\tau$ has a non-empty intersection with both R' and the complement of R'.

If Π is not dense in $Z \cap Y_\tau$, then there exists an open interval K of Z such that K contains no element of Π. Then, by Lemma 3.8, $K \cap Y_\tau$ contains both elements from R' and the complement of R'.

If Π is dense in $Z \cap Y_\tau$, then the result follows from Lemma 3.10. ⊣

PROPOSITION 3.12. *R is not* MSO *definable in M.*

PROOF. The proof is similar to the one of Lemma 3.3. Assume for a contradiction that R is definable in M by some formula $\varphi(x)$ with quantifier depth k. For every $j \in J$ we have $j \in R'$ iff there exists x in $j \cap R$, i.e., iff $T^{k+1}(M_j) \to \exists y \varphi(y)$. The latter property is expressible in the structure N_{k+1}. Therefore R' is definable in the structure N_{k+1} (even with a quantifier-free formula).

Let $n = k + 1$, and let $K \subseteq J$ be an infinite C_n-uniform interval. The set R' is dense and co-dense in every set $S_c = \{a \in K \mid C_n(a) = c\}$ for c in the range of C_n, thus there exist $a_1 \in R' \cap S_c$, and $a_2 \in S_c \setminus R'$. Now K is C_n-uniform, thus there exists an automorphism g of the sub-structure of N_n with domain K which maps a_1 to a_2, and R' is not invariant under g, which contradicts the fact that R' is definable in N_n. ⊣

PROPOSITION 3.13. MSO(M') *is recursive in* MSO(M).

Proof. We show how to reduce $T^n(M')$ to $\mathrm{MSO}(M)$ for every $n \geq 1$.

Let us denote by $(J_n, <)$ the linear order of $\sim_n$-equivalence classes. Let $\pi_n : J \to J_n$ denote the corresponding canonical mapping. For $I \in J_n$ let $\delta_n(I) = \cup_{j \in I} j$. Note that $\delta_n(I)$ is an interval in M. We denote by $O'_{n,I}$ the subchain of M' over $\delta_n(I)$. Observe that

$$O'_{n,I} = \sum_{j \in I} M'_j$$

and

$$M' = \sum_{I \in J_n} O'_{n,I}$$

Let O'_n be the expansion of $(J_n, <)$ by monadic predicates

$$\{Q_{n,\tau'} \mid \tau' \in H_n(\Delta \cup \{R\})\},$$

where Δ is the signature of M, and $Q_{n,\tau'}(I)$ holds iff $T^n(O'_{n,I}) = \tau'$.

The main reduction steps can be represented as follows:

$$T^n(M') \to \mathrm{MSO}(O'_n) \to \mathrm{MSO}(N_n) \to \mathrm{MSO}(M).$$

The first reduction step is a consequence of Theorem 2.7, which shows that the computation of $T^n(M')$ reduces to the one of $T^{f(n,l)}(O'_n)$.

By Lemma 3.5 there is an interpretation of N_n in M, therefore $\mathrm{MSO}(N_n)$ is recursive in $\mathrm{MSO}(M)$.

To complete the proof it is sufficient to show that that $\mathrm{MSO}(O'_n)$ is recursive in $\mathrm{MSO}(N_n)$. This immediately follows from the next Lemma.

Lemma 3.14. *There exists an interpretation of O'_n in N_n. Moreover, there is an algorithm which computes such an interpretation from n.*

Proof. We consider the interpretation map $\mathscr{I} = \pi_n$. The domain $D = J$, the relation $EQ_{\mathscr{I}}$, as well as the ordering relation between $\sim_n$-equivalence classes, are definable in N_n. Thus it remains to provide an interpretation in N_n of each predicate $Q_{n,\tau'}$, i.e., to show that for every n-type τ' in the signature of M', one can define in N_n the predicate $P_{n,\tau'}$ which holds at j iff $(j \in I \in J_n$ and $T^n(O'_{n,I}) = \tau')$.

First note that for every $\tau \in H_n(\Delta)$ there are $\tau^R, \tau^{\neg R} \in H_n(\Delta \cup \{R\})$ such that

$$\tau^R \leftrightarrow (\tau \wedge \forall t R(t)) \text{ and } \tau^{\neg R} \leftrightarrow (\tau \wedge \forall t \neg R(t))$$

Moreover, $\tau^R, \tau^{\neg R}$ are computable from τ.

We claim

$$(6) \qquad P_{n,\tau'}(j) \text{ iff } \begin{cases} \tau' = \tau^R \text{ and } j \text{ is an } n\text{-point such that} \\ \qquad\qquad Type_{n,\tau}(j) \text{ and } j \in R'_{\leq n} \\ \tau' = \tau^{\neg R} \text{ and } j \text{ is an } n\text{-point such that} \\ \qquad\qquad Type_{n,\tau}(j) \text{ and } j \notin R'_{\leq n} \\ \tau' = \mathit{shuffle}(\{\tau^R \mid \tau \in S\} \cup \{\tau^{\neg R} \mid \tau \in S\}), \\ \qquad\qquad \text{where } S \subseteq H_n(\Delta) \text{ and } j \text{ belongs to an } S\text{-interval} \end{cases}$$

Observe that

- Predicates $Type_{n,\tau}$ for $\tau \in H_n(\Delta)$ are in the signature of N_n.
- the set $R'_{\leq n} = \cup_{1 \leq i \leq n} R'_i$ is definable in N_n, by Lemma 3.9.

- for every subset $S = \{\tau_1, \dots, \tau_p\}$ of n-types in the signature of M, the predicate "j belongs to an S-interval of N_n" is definable in N_n.

These observations together with (6) imply that the predicates $P_{n,\tau'}$ are definable in N_n, thus O'_n is interpretable in N_n.

It remains to show that (6) holds.

Assume that j is an n-point. Let I be the $\sim_n$-equivalence class of j. Then j is the only element of I. If $j \in R'_{\leq n}$ then $O'_{n,I}$ is the expansion of M_j by R which holds at every point. Therefore, if $T^n(M_j) = \tau$, then $T^n(O'_{n,I}) = \tau^R$. Hence, (6) holds in this case. The case when j is an n-point and $j \notin R'_{\leq n}$ is similar.

Assume now that j is not an n-point. Then the $\sim_n$ equivalence class I of j is an S-interval for some $S \subseteq H_n(\Delta)$. Hence, I is order-isomorphic to η, the predicates $Type_{n,\tau}$ (for $\tau \in S$) partition I, and each of these predicates is dense in I.

Recall that R' satisfies $(*)$, thus R' is both dense and co-dense in each $Type_{n,\tau} \cap I$ (for $\tau \in S$). If $j \in Type_{n,\tau} \cap I \cap R'$, then $T^n(M'_j) = \tau^R$; if $j \in (Type_{n,\tau} \cap I) \setminus R'$, then $T^n(M'_j) = \tau^{\neg R}$.

Since $O'_{n,I} = \sum_{j \in I} M'_j$, we obtain by Theorem 2.8 that

$$T^n(O'_{n,I}) = shuffle(\{\tau^R \mid \tau \in S\} \cup \{\tau^{\neg R} \mid \tau \in S\}).$$

This completes the proof of (6), of Lemma 3.14 and of Proposition 3.13. ⊣

Third case follows from Proposition 3.12 and Proposition 3.13. ⊣

§4. Further results and open questions. We proved that if the monadic second-order theory of a countable chain C is decidable, then C has a non-trivial expansion with decidable monadic second-order theory.

It would be interesting to obtain a version of our result for first-order logic. However, such a proof requires some new ideas. One obstacle is that there is no first-order formula that expresses that an interval (x, y) is finite. This is expressible in MSO and allowed us to consider three types of intervals.

We also do not know whether the main result of [3] can be extended to first-order logic, namely whether decidability of the first-order theory of a chain which contains an interval of order type ω or $-\omega$ implies that it has non-trivial expansion with decidable first-order theory.

Another interesting issue is to remove the assumption that the ordering is countable and to prove that every chain C has a non-trivial expansion C' such that the monadic theory of C' is recursive in the monadic theory of C. Note that the MSO theory of the real line is undecidable [20].

Acknowledgement. We thank the anonymous referee for useful suggestions.

This research was facilitated by the ESF project AutoMathA. The second author was partially supported by ESF project Games and EPSRC grant.

REFERENCES

[1] A. Bès and P. Cégielski, *Weakly maximal decidable structures*, ***RAIRO Theoretical Informatics and Applications***, vol. 42 (2008), no. 1, pp. 137–145.

[2] ———, *Nonmaximal decidable structures*, ***Journal of Mathematical Sciences***, vol. 158 (2009), pp. 615–622.

[3] A. Bès and A. Rabinovich, *Decidable expansions of labelled linear orderings*, ***Logical Methods in Computer Science***, vol. 7 (2011), no. 2.

[4] A. Blumensath, T. Colcombet, and C. Löding, *Logical theories and compatible operations*, ***Logic and automata: History and Perspectives*** (J. Flum, E. Grädel, and T. Wilke, editors), Amsterdam University Press, 2007, pp. 72–106.

[5] J. R. Büchi, *On a decision method in the restricted second-order arithmetic*, ***Proceedings of the International Congress on Logic, Methodology and Philosophy of Science, Berkeley 1960***, Stanford University Press, 1962, pp. 1–11.

[6] ———, *Transfinite automata recursions and weak second order theory of ordinals*, ***Proceedings of the Internation Congress on Logic, Methodology, and Philosophy of Science, Jerusalem 1964***, North Holland, 1965, pp. 2–23.

[7] J. R. Büchi and C. Zaiontz, *Deterministic automata and the monadic theory of ordinals* ω_2, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 29 (1983), pp. 313–336.

[8] C. C. Elgot and M. O. Rabin, *Decidability and undecidability of extensions of second (first) order theory of (generalized) successor*, this Journal, vol. 31 (1966), no. 2, pp. 169–181.

[9] S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, ***Fundamenta Mathematicae***, vol. 47 (1959), pp. 57–103.

[10] Y. Gurevich, *Modest theory of short chains. I*, this Journal, vol. 44 (1979), no. 4, pp. 481–490.

[11] ———, *Monadic second-order theories*, ***Model-theoretic logics*** (J. Barwise and S. Feferman, editors), Perspectives in Mathematical Logic, Springer-Verlag, 1985, pp. 479–506.

[12] Y. Gurevich, M. Magidor, and S. Shelah, *The monadic theory of* ω_2, this Journal, vol. 48 (1983), no. 2, pp. 387–398.

[13] Y. Gurevich and S. Shelah, *Modest theory of short chains. II*, this Journal, vol. 44 (1979), no. 4, pp. 491–502.

[14] ———, *Interpreting second-order logic in the monadic theory of order*, this Journal, vol. 48 (1983), no. 3, pp. 816–828.

[15] H. Läuchli, *A decision procedure for the weak second-order theory of linear order*, ***Contributions to Mathematical Logic*** (K. Schütte H. A. Schmidt and H.-J. Thiele, editors), North-Holland Publishing Company, 1968, pp. 189–197.

[16] J. A. Makowsky, *Algorithmic uses of the Feferman–Vaught theorem*, ***Annals of Pure and Applied Logic***, vol. 126 (2004), no. 1–3, pp. 159–213.

[17] M. O. Rabin, *Decidability of second-order theories and automata on infinite trees*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[18] A. Rabinovich, *The full binary tree cannot be interpreted in a chain*, this Journal, vol. 75 (2010), no. 4, pp. 1489–1498.

[19] J. G. Rosenstein, ***Linear orderings***, Academic Press, New York, 1982.

[20] S. Shelah, *The monadic theory of order*, ***Annals of Mathematics***, vol. 102 (1975), pp. 379–419.

[21] S. Soprunov, *Decidable expansions of structures*, ***Vopr. Kibern.***, vol. 134 (1988), pp. 175–179, (in Russian).

[22] W. Thomas, *Ehrenfeucht games, the composition method, and the monadic theory of ordinal words*, ***Structures in Logic and Computer Science, A Selection of Essays in Honor of A. Ehrenfeucht***, Lecture Notes in Computer Science, no. 1261, Springer, 1997, pp. 118–143.

[23] ———, *Languages, automata, and logic*, ***Handbook of formal languages*** (G. Rozenberg and A. Salomaa, editors), vol. III, Springer, 1997, pp. 389–455.

[24] ———, *Model transformations in decidability proofs for monadic theories*, ***CSL 2008*** (Michael Kaminski and Simone Martini, editors), Lecture Notes in Computer Science, vol. 5213, Springer, 2008, pp. 23–31.

[25] R. S. Zeitman, ***The composition method***, Ph.D. thesis, Wayne State University, Michigan, 1994.

UNIVERSITY OF PARIS-EST CRÉTEIL
LABORATOIRE D'ALGORITHMIQUE, COMPLEXITÉ ET LOGIQUE
FACULTÉ DES SCIENCES ET TECHNOLOGIE
61 AVENUE DU GÉNÉRAL DE GAULLE
94010 CRÉTEIL, FRANCE
E-mail: bes@u-pec.fr

SACKLER FACULTY OF EXACT SCIENCES
THE BLAVATNIK SCHOOL OF COMPUTER SCIENCE
TEL AVIV UNIVERSITY
RAMAT AVIV 69978
TEL AVIV, ISRAEL
E-mail: rabinoa@post.tau.ac.il