Information and Computation 206 (2008) 1425–1442

Contents lists available at ScienceDirect

Information and Computation

journal homepage: www.elsevier.com/locate/ic

Decidable metric logics

Yoram Hirshfeld *, Alex Rabinovich

Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 15 July 2007
Revised 8 May 2008
Available online 11 October 2008

ABSTRACT

The common metric temporal logic for continuous time were shown to be insufficient, when it was proved that they cannot express a modality suggested by Pnueli. Moreover no finite temporal logic can express all the natural generalizations of this modality. It followed that if we look for an optimal decidable metric logic we must accept infinitely many modalities, or adopt a different formalism. Here we identify a fragment of the second order monadic logic of order with the "+1" function, that expresses all the Pnueli modalities and much more. Its main advantage over the temporal logics is that it enables us to say not just that within prescribed time there is a point where some punctual event will occur, but also that within prescribed time some process that starts now (or that started before, or that will start soon) will terminate. We prove that this logic is decidable with respect to satisfiability and validity, over continuous time. The proof depends heavily on the theory of compositionality. In particular every temporal logic that has truth tables in this logic is automatically decidable. We extend this result by proving that any temporal logic, that has all its modalities defined by means more general than truth tables, in a logic stronger than the one just described, has a decidable satisfiability problem. We suggest that this monadic logic can be the framework in which temporal logics can be safely defined, with the guarantee that their satisfiability problem is decidable.

1. Introduction

The main model for the progression of time is the ordered real line or the non-negative real line with 0 as a first moment of interest. In computer science the latter is commonly assumed. The model for the evolving of a system in *continuous time* is a function f from R^+ to a finite set of propositions (about the system) where $f(t)$ is the set of propositions that hold at the moment t. This is called a *signal*. Alternatively we may think of every proposition as a one place predicate $P(t)$ which holds or not at time t. The mathematical logic that deals with signals is *Monadic Logic of Order* (*MLO*), and its derivatives, the different temporal logics: If the modalities of a temporal logic are defined in a predicate logic, then the temporal logic becomes another way to handle a fragment of the predicate logic. Thus the modalities **U** (for "until") and **S** (for "since") are definable in first order *MLO*, so that the temporal logic *TL*(**U**,**S**) describes a fragment of first order *MLO*. This makes the classical predicate logic a framework in which temporal logics can be compared, and a source of knowledge that can be directly applied or adapted to temporal logic. In Computer Science, systems evolving in time were first modeled as evolving in discrete steps. The corresponding time model was the set of positive integers and the model for the system was a sequence of values stating which predicate holds at any point in the time sequence. The logic that was applied very successfully was mainly temporal logic. A. Pnueli, who was the first to use it and was the main driving force behind its success, was aware of the important connection between temporal logic and classical predicate logic, and he based his work on Kamp's theorem

* Corresponding author.
E-mail address: joram@post.tau.ac.il (Y. Hirshfeld).

0890-5401/$ - see front matter
doi:10.1016/j.ic.2008.08.004

[17], which states that the temporal logic *TL*(**U**,**S**) is equivalent to first order *MLO*. In [8] Pnueli with three coauthors reproved this theorem.

For the last 25 years logics for continuous time were developed, adding a notion of distance, and aiming for a language that is expressive and yet decidable and easy to handle. Temporal logic was enriched and the models chosen for the evolution of systems in continuous time were sequences in which each point was decorated either by a real value ("point sequence models") or by a pair of real values ("interval sequence models"). A very incomplete list of papers where the theory was developed is [5,2,3,18,22,1,10,11]. In particular T. Henzinger with various collaborators developed and researched over the years the temporal logic *MITL* and its derivatives.

In [12,13] we took an alternative approach, using the monadic logic of order as the logical framework, and the positive real line as the model. We were motivated by the belief that the real line is the most natural model for continuous time, that the general theory should apply to all signals, delaying the special treatment of signals with finite variability to a later stage, and that the tools developed for general logic will be useful. In [13] this approach produced a simple temporal logic, which we called *QTL* (for "quantitative temporal logic"). *QTL* was equivalent to *MITL*, it applied to general signals, and it allowed for a generalization of Kamp's theorem as follows: Add to the formation rules of the monadic logic of order the clause: If φ is a formula with s its only free variable then $(\exists s)_{>t}^{<t+1}\varphi$ and $(\exists s)_{>t-1}^{<t}\varphi$ are also formulas. The resulting logic *QMLO* with its natural interpretation is equivalent to *QTL*. The proofs used standard technics and results from general logic (instead of Automata Theory tools). The details were worked out in [15].

An indication that *MITL* (or *QTL*) was not the ultimate decidable metric logic came from "Pnueli's conjecture" (we did not find its original statement): Pnueli defined the modality "within one unit of time X and then Y will occur" and conjectured that it could not be expressed in the logics that were discussed above. In [12,16] we proved that this conjecture was true, and that in fact there was a whole hierarchy of natural decidable extensions of the language such that each is more expressible than the previous one. Moreover: no decidable temporal logic with finitely many modalities contains all these extensions so that any decidable temporal logic which is based on finitely many modalities is incomplete in expressive power. It may be that a temporal logic with infinitely many modalities will be worthwhile considering, provided it is still simple enough, and with maximal expressive power, according to some relevant measure. But which infinite temporal logic should we choose, and by which measure should we evaluate its expressive power?

To try to deepen our understanding we concentrated on classical predicate logic. We looked for a fragment of the monadic logic of order with the $+1$ function (to express nearness) that is expressive yet decidable. The key feature that we considered was to allow to prefix the metric quantifier $(\exists s)_{>t+m}^{<t+n}$ to a formula where both t and s may be free. As we shall see the logic *Q2MLO* has significantly more expressive power than the logics *QTL*, *MITL* and *QMLO*. *Q2MLO* was first discussed in [12] and the main result of this work was announced and discussed there.

Using a "compositional method" we prove that *Q2MLO* is decidable. In particular, every temporal logic that has its modalities defined in *Q2MLO* is automatically decidable. For temporal logics we have an even stronger result. We denote by $Q2MLO_\exists$ the logic that allows existential set quantifiers in front of *Q2MLO* formulas, and we show that every temporal logic that is based on modalities which are defined in $Q2MLO_\exists$ is decidable. The proof is quite general, and it applies to all logics that obey compositionality. It gives an algorithm that reduces satisfiability questions in a logic enriched by existential metric quantifiers, to the original logic.

We cannot "prove" that $Q2MLO_\exists$ is the ultimate decidable metric logic, but it seems on the one hand very expressive and on the other hand, very small extensions render it undecidable. Such extensions are allowing to prefix $(\exists s)_{>t+m}^{<t+n}$ in front of a formula with two variables, where only one is t or s, or allowing trivial statements on the whole interval $(t,t+1)$.

Important work along similar lines was previously done by Wilke [22] who specified a fragment of the second order monadic logic of order with distance, and showed that this logic is decidable for point sequence models, and that it is expressive enough to allow the definition of different temporal logics. It is difficult to compare Wilke's work with the current one; on the one hand his results are proved for point sequence models, for which everything that happens has a first point where it happens. This is clearly not the general case. On the other hand his proofs are based on automata theory, and cannot be extended to general signals. For the real line Wilke's logic is more expressive than our $Q2MLO_\exists$ but it is also undecidable by [20], since it includes all of (the non-metric) second order *MLO*. Wilke's logic is probably decidable for the real line with predicates of finite variability, but his proofs do not adapt straightforwardly to the real line model.

The paper is divided as follows: In Section 2 we describe the background and previous results. In Section 3 we define the new logics *Q2MLO* and $Q2MLO_\exists$ and prove the main result, that the logics are decidable. Section 4 extends the decidability result to temporal logics whose modalities are second order functionals defined by $Q2MLO_\exists$ formulas. Section 5 shows how this method can be used to define new decidable modalities. Admittedly, with the possible exception of Pnueli's modalities they do not seem very useful. Section 6 exemplifies that what seems like minor modifications in the definitions renders the logics undecidable. Finally Section 7 explains that the methods used are abstract and general enough to apply to any logic that satisfy appropriate compositional requirements.

The paper is theoretical in nature, and does not suggest any pragmatical algorithms. We show that a very general framework is decidable, by supplying the link that reduces the problem to the non-metric case. Since decidability of satisfiability and validity is non-elementary for standard monadic logic of order the question whether the reductions are elementary is not very important, and indeed the natural way to construct the disjunction of types that replaces any formula is non-elementary.

2. Monadic logic of order and simple quantitative logics of order

In this section we summarize the part of our previous work which is relevant to the discussion. We recall the definitions of the monadic logic of order (*MLO*) and the temporal logic (*TL*). We then repeat the definitions from [14] of their quantitative expansions *QMLO* and *QTL*.

We cite the theorems from [15] concerning the decidability and complexity of these extensions, and we recall the definitions of *Timer normal form logics*, which are (up to the addition of existential quantifiers) simple formulas of the given quantitative logic. In the cases that we investigated in [15] a formula of quantitative logic was reduced by an effective, satisfiability preserving transformation to timer normal form. The main technical result was a similar reduction of a timer normal form formula to formula of pure (non-metric) monadic logic of order. This reduced the decidability and complexity questions for quantitative logics to the similar questions about the non-metric logics. The last subsection introduces the compositional methods that will allow us to similarly reduce the stronger logics discussed in this work.

2.1. Monadic logic of order (MLO)

The syntax of MLO has in its vocabulary *individual (first order) variables* $t_0, t_1, \ldots$, *unary predicate names* $X_0, X_1, \ldots$ and one binary relation $<$ (the order). *Atomic formulas* are of the form $X(t)$, $t_1 < t_2$ and $t_1 = t_2$. *MLO formulas* are obtained from atomic formulas using the Boolean connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$. We will freely use derived notations like $x > y$ for $y < x$ or $x \le y$ for $y < x \vee x = y$. In *Second order MLO* a particular logic is declared when we decide which of the predicate names above is a *predicate constant* and which is a *predicate variable*. Formulas may have second order quantifiers $\exists X$ and $\forall X$, where X is a predicate variable. As usual if φ is a formula we may write $\varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$ to indicate that the free variables in φ are among $t_1, \ldots, t_k$ and $X_1, \ldots, X_m$.

A *(canonical real time) model* is the non-negative real line R^+ with its natural order and some unary predicates, corresponding to the predicate constants. We shall not repeat the inductive definition saying when is a formula satisfied. Recall that in order to check if the formula $\varphi(t_1, \ldots t_k; X_1, \ldots, X_m)$ is true we need to specify what are the elements $\tau_1 \ldots \tau_k$ in R^+ and the predicates (subsets) $P_1, \ldots, P_m$ over R^+ which are assigned to the variables $t_1, \ldots, t_k, X_1, \ldots, X_m$. Hence the notation will usually be

$$\langle R^+, \tau_1, \ldots, \tau_k; P_1, \ldots, P_m \rangle \models_{MLO} \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$$

which we also abbreviate to $R^+ \models \varphi[\tau_1, \ldots, \tau_k; P_1, \ldots, P_m]$ or even to $R^+ \models \varphi[\overline{\tau}, \overline{P}]$ where the bar denotes a tuple of appropriate length. When we define the semantics of a second order formula or when we deal with validity and satisfiability of a first order formula it is necessary to specify over which predicates should the variables X range. In *standard MLO* they range over all unary predicates (i.e. – subsets). A requirement that is often imposed in the literature is that in every bounded time interval a system can change its state only finitely many times. This requirement is called *finite variability* (or non-Zeno) requirement. We consider also finite variability interpretations of first-order and second-order MLO. Under these interpretations monadic predicates range over predicates with finite variability. We note that decidability results for general signals, in predicate logics extending *MLO* or in temporal logics extending *TL*(**U**,**S**) hold also for the same languages, where the unary predicates range only on sets with finite variability. This is so because over the real line, or over the positive reals, finite variability is definable in the language, due to the Bolzano Weierstrass theorem which assures that a bounded set of infinitely many points has a point of Accumulation. A predicate X is of infinite variability if and only if:

$$\exists t (Llim(X,t) \wedge Llim(\neg X,t)) \vee (Rlim(X,t) \wedge Rlim(\neg X,t))$$

where $Llim(X,t)$ is $\forall t_1 \exists t_2 (t_1 < t) \rightarrow (t_1 < t_2 < t \wedge X(t_2))$ and $Rlim(X,t)$ is $\forall t_1 \exists t_2 (t_1 > t) \rightarrow (t_1 > t_2 > t \wedge X(t_2))$.

Therefore a formula holds in the model with finite variability predicates if and only its modification, that requires every predicate to be of finite variability, holds in the model with general predicates.

2.2. Temporal logics and truth tables

Temporal logics uses logical constructs called *"modalities"* to create a language that is free from variables and quantifiers. Here is the general logical framework to define temporal logics:

The syntax of the temporal logic $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)})$ has in its vocabulary *monadic predicate variables* $X_1, X_2, \ldots$ and a sequence of *modality names* with prescribed arity, $O_1^{(k_1)}, \ldots, O_n^{(k_n)}$ (the arity notation is usually omitted). The formulas of this temporal logic are given by the grammar:

$$\varphi ::= X | \neg\varphi | \varphi \wedge \varphi | O^{(k)}(\varphi_1, \ldots, \varphi_k)$$

When the particular modality names are unimportant or they are clear from the context we omit them and write *TL* instead of $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)})$.

Structures for TL are again linear orders with monadic predicates $M = \langle A, <, P_1, P_2, \ldots, P_n, \ldots \rangle$, where the predicate P_i isassigned to a predicate constant X_i. Every modality $O^{(k)}$ is interpreted in every structure M as an operator $O_M^{(k)} : [\mathcal{P}(A)]^k \rightarrow$

$\mathcal{P}(A)$ which assigns "the set of points where $O^{(k)}[S_1,\dots,S_k]$ holds" to the k-tuple $\langle S_1,\dots,S_k\rangle \in \mathcal{P}(A)^k$. (Here $\mathcal{P}$ is the power set notation, and $\mathcal{P}(A)$ denotes the set of all subsets of A.) Once every modality corresponds to an operator the semantics is defined by structural induction:

- For atomic formulas

$$\langle M,\tau\rangle \models_{TL} X \quad \text{iff} \quad \tau \in P, \text{ where the monadic predicate } P \text{ is assigned to } X.$$

- For Boolean combinations the definition is the usual one.
- For $O^{(k)}(\varphi_1,\dots,\varphi_k)$

$$\langle M,\tau\rangle \models_{TL} O^{(k)}(\varphi_1,\dots,\varphi_k) \quad \text{iff} \quad \tau \in O^{(k)}_M(A_{\varphi_1},\dots,A_{\varphi_k})$$

where $A_\varphi = \{\tau : \langle M,\tau\rangle \models_{TL} \varphi\}$ (we suppressed predicate parameters that may occur in the formulas).

Usually we are interested in a more restricted case; for the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets A_{φ_i} of points satisfying φ_i and the set of points $O[A_{\varphi_1},\dots,A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

Truth tables: A formula $\overline{O}(t_0,X_1,\dots X_k)$ in the predicate logic L is a *Truth Table* for the modality $O^{(k)}$ if for every structure M

$$O_M(P_1,\dots,P_k) = \{\tau : M \models_{MLO} \overline{O}[\tau,P_1,\dots,P_k]\}\ .$$

The following proposition justifies our view that a temporal logic is an alternative presentation of fragments of the predicate logic in which it is defined.

Proposition 2.1. *Let PL be a logic that allows substitution of formulas with one free first order variable instead of one place predicates. If every modality in the temporal logic TL has a truth table in the logic PL then to every formula $\varphi(X_1,\dots,X_n)$ of TL there corresponds effectively (and naturally) a formula $\overline{\varphi}(t_0,X_1,\dots X_n)$ of PL such that for every $M,\tau \in M$ and predicates $P_1,\dots,P_n$*

$$\langle M,\tau,P_1,\dots,P_n\rangle \models_{TL} \varphi \quad \text{iff} \quad \langle M,\tau,P_1,\dots,P_n\rangle \models_{MLO} \overline{\varphi}\ .$$

Every modality that we found in the literature has a truth table in predicate logic, and the temporal logic can be thought of as syntactical sugar for a fragment of classical logic. Thus the modality $\Diamond X$, "eventually X", is defined by $\varphi(t_0,X) \equiv \exists t > t_0\, X(t)$. The modality $\overleftarrow{\Diamond} X$, "$X$ has happened before", is defined by $\varphi(t_0,X) \equiv \exists t < t_0\, X(t)$. The modality $X\mathbf{U}\,Y$, "X *until* Y", is defined by

$$\psi(t_0,X,Y) \equiv \exists t_1(t_0 < t_1 \wedge Y(t_1) \wedge \forall t(t_0 < t < t_1 \rightarrow X(t))).$$

The modality $X\mathbf{S}\,Y$, "X *since* Y", is defined by

$$\psi(t_0,X,Y) \equiv \exists t_1(t_0 > t_1 \wedge Y(t_1) \wedge \forall t(t_1 < t < t_0 \rightarrow X(t))).$$

2.3. The temporal logic $TL(\mathbf{U},\mathbf{S})$

We start with the logic, $TL(\mathbf{U},\mathbf{S})$ that has the modalities $\mathbf{U}$ (for "until") and $\mathbf{S}$ (for "since") defined above. Since the two modalities have truth tables in *MLO* this logic describes a fragment of (first-order) *MLO*. The next theorem shows that $TL(\mathbf{U},\mathbf{S})$ is just as expressive (as far as formulas with at most one free time variable are concerned) as *MLO*. We recall the terminology used when comparing languages:

Definition 2.2. Let Σ be a class of structure (possibly with a single structure), and let L and L' be logics each with a fixed interpretation in the class.

(1) L is *expressible* in L' over the class if for every formula φ of L there is a formula φ' of L' such any tuple (of the proper length) in any structure of the class satisfies φ if and only if it satisfies φ'.
(2) L and L' are expressively equivalent over the class if each logic is expressible in the other.

Theorem 2.3 ([17], reproved in [8])**.** *The temporal logic $TL(\mathbf{U},\mathbf{S})$ is expressively equivalent over the two canonical structures, the real line and the natural numbers, to the fragment of first-order MLO of formulas with at most one free (first-order) variable.*

From now on when we say "temporal logic", *TL*, or "pure temporal logic" we mean $TL(\mathbf{U},\mathbf{S})$.
In *TL* we have the following definable operations:

$$\Diamond X \equiv True\ \mathbf{U}\ X$$
$$\Box X \equiv \neg\Diamond\neg X$$
$$\overleftarrow{\Diamond} X \equiv True\ \mathbf{S}\ X$$

$\overleftarrow{\Box} \equiv \neg \overleftarrow{\Diamond} \neg X$

$Always(X) \equiv \overleftarrow{\Box} X \wedge X \wedge \Box X$

"Always" acts like the universal closure of a formula in relational logic, and it turns it into a sentence, which has the same value at any point in the structure.

2.4. Quantitative temporal logic and quantitative monadic logic of order

The logics *MLO* and *TL*(**U**,**S**) are not suitable to deal with quantitative statements like "X will occur within one unit of time". We add to *TL*(**U**,**S**) the modalities $\Diamond_1 X$ (X will happen within the next unit of time) and $\overleftarrow{\Diamond}_1 X$ (X happened within the last unit of time).

Definition 2.4. Quantitative temporal logic, *QTL*, is the temporal logic the temporal logic constructed from **U**, **S** *and* two new modalities $\Diamond_1 X$ and $\overleftarrow{\Diamond}_1 X$ defined by the tables with free variable t_0:

$$\Diamond_1 X: \quad \exists t((t_0 < t < t_0 + 1) \wedge X(t))$$

$$\overleftarrow{\Diamond}_1 X: \quad \exists t((t < t_0 < t + 1) \wedge X(t)) .$$

We also introduce the derived modalities:

$$\Box_1 X = \neg \Diamond_1 \neg X \qquad \overleftarrow{\Box}_1 X = \neg \overleftarrow{\Diamond}_1 \neg X$$

Next we intend to identify the fragment of the monadic logic of order with the $+1$ function that corresponds to *QTL*. This fragment will use the function $t + 1$ only in a very restricted form as indicated in (3) and (4). We use "bounded quantifiers" $(\exists t)_{>t_0}^{<t_0+1}$ and $(\exists t)_{>t_0-1}^{<t_0}$ as shorthand:

$$(\exists t)_{>t_0}^{<t_0+1} \varphi \equiv \exists t(t_0 < t < t_0 + 1 \wedge \varphi(t))$$

We do not have substraction in the language and we define:

$$(\exists t)_{>t_0-1}^{<t_0} \varphi \equiv \exists t((t < t_0 < t + 1) \wedge \varphi(t))$$

Note that for points t_0 smaller than 1 it just means that there is some previous t satisfying φ.

Definition 2.5. Quantitative monadic logic of order, *QMLO* , is the fragment of the monadic logic of order with the $+1$ function, which is built from the atomic formulas $t < s$, $t = s$, $X(t)$, where t and s are first order variables, using Boolean connectives, first order quantifiers and the following rule: if $\varphi(t)$ is a formula of *QMLO with t its only first order free variable* then $(\exists t)_{>t_0}^{<t_0+1} \varphi$ and $(\exists t)_{>t_0-1}^{<t_0} \varphi$ are formulas of *QMLO*.

Theorem 2.6. *QTL is expressively equivalent to the fragment of QMLO of formulas with at most one free (first-order) variable.*

Is the language strong enough to deal with intervals of length larger than one?

For *QMLO* and *QTL* the answer is positive. This is the content of the next theorem. We use notations like $(\exists t)_{>t_0+n}^{<t_0+n+m}$, $(\exists t)_{>t_0+n}^{<\infty}$, $(\exists t)_{\geq t_0+n}^{\leq t_0+n+m}$ which are self explanatory. With these notations we have:

Theorem 2.7. *The extension L of QMLO by the following rules is expressive equivalent to QMLO over the canonical model: if $\varphi(t)$ is an L formula with the only free variable t then the following are L formulas:*

(1) $(\exists t)_{>t_0+n}^{<t_0+n+m} \varphi(t)$, *where n is an integer (possible negative) and m a positive integer.*
(2) $(\exists t > t_0 + n)\varphi(t)$ *(denoted also by* $(\exists t)_{>t_0+n}^{<\infty} \varphi(t)$*) and* $(\exists t < t_0 + n)\varphi(t)$ *(denoted by* $(\exists t)_{>-\infty}^{<t_0+n} \varphi(t)$*), where n is an integer.*
(3) *Similar to (1) or (2) above with weak inequality replacing one or both occurrences of the strong inequality.*

The theorem was proved in [12,15].

2.5. Decidability and complexity for quantitative logics

The main results of [13,14] were the following:

Theorem 2.8 (*Decidability*)**.**

(1) *Satisfiability in the canonical model is decidable for QMLO and for QTL.*

(2) *Satisfiability in the finite variability canonical model is decidable for QMLO and for QTL*.

Theorem 2.9 (*PSPACE complexity*)**.**

(1) *The satisfiability problem for QTL in the finite variability canonical model is in PSPACE*.
(2) *The satisfiability problem for QTL the canonical model is in PSPACE*.

2.6. Timer normal form

The proof of the decidability and complexity theorems is based on two reduction steps: First metric formulas are reduced to simple metric formulas called *formulas in timer normal form*, then a heavy technical theorem reduces a timer normal form formula to a non-metric formula. This reduces the decidability and complexity of metric formulas to the similar question on non-metric formulas. Timer normal form plays a central role also in this work and we will define it again. The definition will slightly differ from the definition in [15]. The modified definition reflects better the idea of the timer, and is easier to work with in this paper. A timer for a formula $\varphi(t)$ is a new unary predicate $C(t)$ which is true at points where φ was true uninterruptedly for the last unit of time. This is a special case of an auxiliary unary predicate which can be described by a formula.

We start with the definition that was used in [13]:

Definition 2.10 (*Timer formulas*)**.** We call the following *QMLO* formula *a simple Timer Formula*

$$\mathrm{Timer}(X,Y) \equiv \forall t(Y(t) \longleftrightarrow (\forall t_1)^{<t}_{>t-1} X(t_1))$$

we say that Y is *a timer for* X.

For every n and $2n$ distinct variables $X_1,\dots,X_n,Y_1,\dots,Y_n$ we define the formula:

$$\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \equiv \bigwedge_i \mathit{Timer}(X_i,Y_i)$$

In the formula $\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n)$ each Y_i is a timer that measures if X_i persisted for the duration of the last unit of time.

Definition 2.11 (*Timer Normal Form*)**.** A formula is said to be in first-order (second-order, or temporal) *timer normal form* if it is of the form

$$\exists \overline{W}(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \wedge \varphi),$$

where $\overline{W}$ is a list of monadic variables (among those explicit in Timer_n or implicit in φ), and φ is first order *MLO*, second-order *MLO* or *TL*(**U**,**S**) formula, respectively.

Theorem 2.12 (*Reduction to Timer Normal Form*)**.** *Every formula can be brought to Timer normal form*:

(1) *There is an algorithm which associates with any formula $\varphi(\bar{t},\overline{Z})$ of QMLO , a formula $\overline{\varphi}(\bar{t},X_1,\dots,X_n,Y_1,\dots,Y_n,\overline{Z})$ of* (*non-metric*) *first-order MLO*,
such that φ is equivalent to the formula:

$$\exists\overline{X}\exists\overline{Y}\Big(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \wedge \overline{\varphi}\Big)$$

In particular, φ is satisfiable iff the following formula is satisfiable:
$\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \wedge \overline{\varphi}$

(2) *There is an algorithm which associates with any QTL formula $\varphi(\overline{Z})$, a formula $\overline{\varphi}(X_1,\dots,X_n,Y_1,\dots,Y_n,\overline{Z})$ of TL*(**U**,**S**)), *such that φ is equivalent to the formula*:

$$\exists\overline{X}\exists\overline{Y}\Big(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \wedge \overline{\varphi}\Big)$$

In particular, φ is satisfiable iff the following formula is satisfiable:

$$\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n) \wedge \overline{\varphi}$$

The main theorem in [15] eliminates the metric altogether:

Theorem 2.13 (*Elimination of the metric*)**.** *There is an algorithm which associates with every n a formula $\overline{\mathrm{Timer}_n}$ of pure monadic logic MLO in the same $2n$ set variables as Timer_n, such that*:

(1) *If the predicates $P_1,\dots,P_n,Q_1,\dots,Q_n$ over R^+ satisfy the formula Timer_n then they satisfy the formula $\overline{\mathrm{Timer}_n}$*.

(2) *If the predicates* $P_1,\ldots,P_n,Q_1,\ldots,Q_n$ *over* R^+ *satisfy the formula* $\overline{\text{Timer}}_n$ *then there is an order preserving bijection* $\rho : R^+ \to R^+$ *such that*
$P_1\rho,\ldots,P_n\rho,Q_1\rho,\ldots,Q_n\rho$ *satisfy* Timer_n
($P_i\rho$ *is the image of* P_i *under the bijection* ρ*)*

The last theorem entails the desired result:

Theorem 2.14. *For every first-order or second-order monadic formula or temporal* φ :

$\overline{\text{Timer}}_n \wedge \varphi$ is satisfiable iff $\text{Timer}_n \wedge \varphi$ is satisfiable.

The satisfiability problem for first order monadic logic of order is decidable by [4]. Since $\overline{\text{Timer}_\text{n}}$ is a first-order *MLO* formula, we have:

Theorem 2.15. *The satisfiability problem for formulas in first-order timer normal form is decidable with respect to the class of all structures, and with respect to the class of structures with finite variability.*

For the class of structures with finite variability it was shown in [19] that second order monadic logic of order was decidable. Therefore we have a stronger decidability result for this class:

Theorem 2.16. *The satisfiability problem for formulas in second order timer normal form is decidable with respect to the class of structures with finite variability.*

2.7. The Pnueli modalities

We examine a statement that was suggested by A. Pnueli, in order to check whether logics of the type of *QTL* are strong enough to express the natural queries that come up when we describe properties of a program. We denote by $P_2(X,Y)$ ("P" for Pnueli) the modality which says that X and then Y will be true at two points within the next unit of time. Its truth table is $(\exists t)^{<t_0+1}_{>t_0}.\exists t_1(t_0 < t_1 < t \wedge X(t_1) \wedge Y(t))$. Pnueli conjectured that the statement cannot be expressed in the metric temporal languages that we discussed up to now. To investigate this question we introduced the sequence of the *Pnueli modalities*, $P_n(X_1,\ldots,X_n)$, for every natural number. $P_n(X_1,\ldots,X_n)$ says that there is a sequence of points $t_1 < \cdots < t_n$ within the next unit interval of time, such that for $i = 1,\ldots,n$, X_i holds at t_i. We also introduced the *counting modality* $C_n(X)$, that says that X will be true at least at n points in the next unit interval of time. That is, $C_n(X)$=$P_n(X,\ldots,X)$. With these definitions we proved in [16]:

Theorem 2.17

(1) *For every natural number n the modality* $C_{n+1}(X)$ *is not expressible in the temporal logic* $TL(\mathbf{U},\mathbf{S},P_1,\ldots,P_n)$.
(2) *There is no temporal logic with finitely many modalities that can express the modalities* C_n *for all natural n.*

Some remarks are in order:

(1) The last claim must be made precise. We must state in which formal framework there is no finite temporal logic. This is done in [16].
(2) Kamp's theorem says that in the non-metric case there is a simple temporal logic with two modalities that is expressibly complete, and therefore as expressive as can be hoped for. The last theorem shows that there is no hope for a simple extension to Kamp's theorem: Any temporal logic with claim to full expressiveness will must have infinitely many modalities.
(3) The logic obtained by adding to *QTL* all the connectives P_n is decidable, yet this does not follow from any previous result and it must be proved directly. The same applies to any proper extension of *QTL*, which is decidable. As will be seen, our main result will make all these special proofs unnecessary.

2.8. Elements of composition method

Families of structures of the same type may be combined in different ways to compose a new structure of the same type. The "compositional method" applies when a structure is composed from simpler structures, and the theory of the composite structure can be reduced to the theory of its components. Ehrenfeucht used it in [6] for ordered structures, and our proofs follow his steps. The method was systematically developed and used by Feferman-Vaught [7], Shelah [20] and others, and the subject is surveyed in [9,21]).

In the next section, in the proofs of the main theorems, we use this method repeatedly, decomposing intervals into subintervals and expressing a statement about the interval in terms of statements about the subintervals. This is done twice when we show that all the formulas of *Q2MLO* can be expressed using only quantifiers $(\exists s)^{<t+1}_{>t}$ and $(\exists s)^{<t}_{>t-1}$ (Theorem 3.8). Then it will be done again in the Main Theorem 3.9, to entirely eliminate the metric.

We formulate the theorems that we need for the special case where only finitely many models are involved and the composition is a serial composition of linearly ordered models.

For every non-negative integer m we consider structures of the form $M = (A, <, S_1, \ldots, S_m)$, where $<$ is a linear order on a set A and S_i are subsets of A. Let L be the first order predicate logic with the signature $\{<, S_1, \ldots, S_m\}$. We recall the notion $qd(\varphi)$ of *quantifier depth* of a formula:

$qd(\varphi) = 0$ if φ is atomic, $qd(\neg\varphi) = qd(\varphi)$, $qd(\varphi \vee \psi) = max\{qd(\varphi), qd(\psi)\}$, and $qd(\exists t\varphi) = qd(\varphi) + 1$.

From here up to the end of the subsection we fix m, and all the structures considered have a linear order and m unary predicates. We start with the following definition:

Definition 2.18. A formula $\psi(t_1, \ldots, t_n)$ with at most n free variables as indicated, and with quantifier depth at most k is called a (k,n)-*type* or a (k,n) *Hintikka formula* if every formula $\theta(t_1, \ldots, t_n)$ with at most n free variables as indicated, and with quantifier depth at most k either follows from ψ or it contradicts ψ.

A basic property of the logic is that there are finitely many (k,n)-types for every k,n, that every formula is equivalent to a disjunction of (k,n)-types, and that the process that associates with a formula the equivalent disjunction of types is effective. Formally:

Proposition 2.19. *For every n and k there is a number $\tau = \tau(k,n)$, and τ (k,n)-types, $\psi_1(t_1, \ldots, t_n), \ldots, \psi_\tau(t_1, \ldots, t_n)$ such that*:

(1) *Every formula $\theta(t_1, \ldots, t_n)$ with quantifier depth at most k is equivalent to a formula $\overline{\theta}(t_1, \ldots, t_n)$ which is a disjunction of (k,n)-types.*
(2) *There is an algorithm that given k,n computes $\tau(k,n)$, and the (k,n)-types $\psi_1(t_1, \ldots, t_n), \ldots, \psi_\tau(t_1, \ldots, t_n)$, and which given a formula $\theta(t_1, \ldots, t_n)$ with quantifier depth at most k computes the equivalent disjunction $\overline{\theta}(t_1, \ldots, t_n)$ of (k,n)-types.*

(Note that this general method to deal with formulas is not efficient in the sense of complexity theory, and the algorithm is non-elementary.)

The last proposition can be stated slightly differently: Two structures M,M' are called k-equivalent (written: $M \equiv_k M'$) if $M \models \varphi \Leftrightarrow M' \models \varphi$ for every sentence φ of quantifier depth k. This is an equivalence relation on structures, and each (k,0)-type in the proposition is the complete k-theory of one of the equivalence classes (expressed as a single sentence).

Definition 2.20

Let M_0,M_1 be two structures in the same monadic logic of order (i.e., interpreting the same unary predicate names), and with disjoint domains. Their *concatenation* or *ordered sum* $M_0 + M_1$ is defined as follows: The domain of $M_0 + M_1$ is the union of the domains of M_0 and M_1 the interpretation of a unary predicate S is the union of its interpretation in M_0 and in M_1, all elements of M_0 are less than all elements of M_1 and if two elements are in the same M_i, then their order in $M_0 + M_1$ remains as it was in M_i.

The composition theorem for ordered sums is the following:

Theorem 2.21 (*Composition Theorem*)**.** *The $(k,0)$-types of M_0 and M_1 determine the $(k,0)$-type of the ordered sum $M_0 + M_1$: For every sentence φ of quantifier depth k there is a finite, effectively determined, sequence of pairs of sentences of quantifier depth k, $\langle\psi_1,\theta_1\rangle, \cdots, \langle\psi_q,\theta_q\rangle$ such that for every pair of structures M_0 and M_1, $M_0 + M_1 \models \varphi$ if and only if for some $i \le q$, $M_0 \models \psi_i$ and $M_1 \models \theta_i$.*

The theorem extends by induction to any finite sum of structures.

The theorem assumes that two or more structures are combined to form a new structure. We want to apply it to the case that we decompose a larger structure into two or more substructures. We want to replace the formulas ψ_i and θ_i that speak about the components as separate structures, into formulas ψ'_i and θ'_i that speak about these components inside the larger structure. We will recall one more notion from model theory, that of *relativizing a formula*. We stick to the case of an ordered structure, and the relativization of the formula to a segment of the structure.

Let L be any first order language that includes a binary symbol $<$ and let x,y be two variables. For any formula ϕ not mentioning x and y we define by induction its *restriction* ϕ' to the interval $[x,y]$: For atomic formulas $\phi' = \phi$, $(\neg\phi)' = \neg(\phi')$, $(\psi \vee \phi)' = \psi' \vee \phi'$, and most important: $(\exists t\phi)' = (\exists t[(x \le t \le y) \wedge \phi'])$.

Note that x and y become free variables in ϕ'. With these notations we have:

Proposition 2.22. *Let L be a first order monadic language of order, let $\phi(t_1,\ldots,t_r)$ be a formula with the free variables as indicated, and let ϕ' be its restriction to $[x,y]$. Let M be a structure for the language, $a < b$ elements in M, and let M' be the substructure whose universe is the interval $[a,b]$ of the universe of M and let $a \le a_1 < \cdots < a_r \le b$ Then*:

$$M' \models \phi(a_1,\ldots,a_r) \quad \textit{iff} \quad M \models \phi'(a_1,\ldots,a_r,a,b)$$

where a and b corresponds to the two extra variables x and y of ϕ'.

This does not conclude the preparation for the composition theorem, in the form that suits us. We must define the relativization of a formula to an unbounded (in the structure) subinterval $[x,\infty)$, which adds only one new free variable, x, and the unbounded (in the structure) interval $(-\infty,a]$. We must modify the notion of relativization, so that $M'' \models \phi(a_1,\ldots,a_r)$ iff $M \models \phi''(a_1,\ldots,a_r,a,b)$, where M'' is the substructure whose universe is the half open interval $[a,b)$. We leave it to be completed by the reader.

To avoid double indexing we state the next theorem for two variables and not for the general case.

Theorem 2.23 (*Second Composition Theorem*)**.** *For every k there is an effectively determined finite sequence of quadruples of formulas*

$$\langle\phi_1(x),\psi_1(x,y),\theta_1(x,y),\chi_1(y)\rangle,\cdots,\langle\phi_q(x),\psi_q(x,y),\theta_q(x,y),\chi_q(y)\rangle$$

such that for $i = 1,\ldots,q$, $\phi_i(x)$ is relativized to the interval $(-\infty,x)$, $\psi_i(x,y)$ and $\theta_i(x,y)$ are relativized to the interval $[x,y)$, and $\chi_i(y)$ is relativized to the interval $[y,\infty)$, and such that:

For every formula $\varphi(x,y,z)$ of quantifier depth k with three free variables there is a set $A_\varphi \subseteq \{1,\ldots,q\}$ (computable from φ) such that

$$(x < y < z) \wedge \varphi(x,y,z) \equiv (x < y < z) \wedge \bigvee_{i \in A_\varphi} (\phi_i(x) \wedge \psi_i(x,y) \wedge \theta_i(y,z) \wedge \chi_i(z))$$

With $\alpha_i(x,z) = \phi_i(x) \wedge \psi_i(x,z)$ and $\beta_i(x,z) = \theta_i(z,y) \wedge \chi_i(y)$ we get the following version:

Theorem 2.24 (*Decomposition Theorem*)**.** *For every formula $\varphi(x,y)$ there is a finite set of pairs (computable from φ)*

$$\langle\alpha_1(x,z),\beta_1(z,y)\rangle,\cdots\langle\alpha_q(x,z),\beta_q(z,y)\rangle$$

such that for every structure M and for any $a < b$ in M, the following are equivalent:

(1) $M \models \varphi(a,b)$.
(2) *For some point c such that $a < c < b$, $M \models \bigvee_{i=1}^{q}(\alpha_i(a,c) \wedge \beta_i(c,b))$.*
(3) *For every point c, such that $a < c < b$, $M \models \bigvee_{i=1}^{q}(\alpha_i(a,c) \wedge \beta_i(c,b))$.*

3. The logic Q2MLO

The results of the last section (see 2.7) indicate that there is no finite metric temporal logic that is maximal in a natural sense among the decidable temporal logic. This led us to abandon the direct temporal logic approach and focus again on predicate logic. We know that the class of formulas in timer normal form is a decidable class, and we used this fact to prove that *QMLO* is decidable. We will prove that a stronger predicate logic is reducible to timer normal form and is therefore decidable. In this section we analyze this logic *Q2MLO* and prove that it is reducible to timer normal form, and hence decidable.

Definition 3.1

(1) *Q2MLO* is the predicate logic that has atomic formulas $t = s$, $t < s$ and $X(t)$, is closed under Boolean connectors and first order quantifications, and under the "metric quantifiers":
If $\varphi(t_0,t)$ is a formula in *Q2MLO* with t and t_0 its only free first-order variables and $m < n$ are integers then $(\exists t)_{>t_0+m}^{<t_0+n}\varphi(t_0,t)$ is a formula of *Q2MLO* (in the free variable t_0).
(2) The sublogic *Q2MLO*$_0$ of *Q2MLO* is defined as follows: A formula of *Q2MLO* is in *Q2MLO*$_0$ if all its metric quantifiers are of the form: $(\exists t)_{>t_0}^{<t_0+1}$ or $(\exists t)_{>t_0-1}^{<t_0}$.

Remark 3.2

(1) The difference between *Q2MLO* and *QMLO* may seem small, but it is not. The fact that in *Q2MLO* the formula φ that follows the metric quantifier is allowed to have two free variables, enhances the expressibility so that we can say: "within one unit of time a process that starts now (or that started before, or that will start in the near future) will be completed". It also changes the nature of the logic. The *QMLO* metric quantifier corresponds straightforwardly to a temporal modality as

a result of the fact that it is applied to a formula with one free variable, which speaks of one point in time. This is no more the case with *Q2MLO*, as the quantifier is applied to the formula $\varphi(t_0,t)$ which speaks about two points (or equivalently, about the interval (t_0,t)).

(2) We will show that $Q2MLO_0$ is as expressive as *Q2MLO*, so that it might be more natural to take $Q2MLO_0$ as the basic object, similarly to the definition of *QMLO*. We defined *Q2MLO* as the basic object to be consistent with the way that it was defined in [12], where it was first introduced.

3.1. Expressive equivalence of Q2MLO and $Q2MLO_0$

The proof that $Q2MLO_0$ is as expressive as all of *Q2MLO* uses heavily composition methods. We divide the proof into two propositions. The first one deals with weak inequalities in the metric quantifiers, like $(\exists s)^{\leq t+1}_{>t}$ and the second deals with more general intervals of length 1, like $(\exists s)^{\leq t+n+1}_{>t+n}$.

Remark 3.3 (*On the algorithmic nature of our proofs*)**.** All our proofs are constructive. We often state "for every formula φ in a language L_1 there exists an equivalent formula ψ in a language L_2". However, from the proofs it will follow not only that such ψ exists, but that there exists an algorithm which constructs for every $\varphi \in L_1$ an equivalent formula $\psi \in L_2$.

On the other hand the proofs use the composition theorems to reduce formulas from *Q2MLO* and other complex logics to formulas of standard *MLO*, and the composition theorems produce an algorithm which is not even elementary (we recall that if we define the function $f_m(n)$ by induction on m so that $f_1(n) = 2^n$, and $f_{m+1}(n) = 2^{f_m(n)}$ then a non-elementary function of n is not bounded by any of the functions $f_m(n)$). There may be more efficient reductions that are elementary, but they cannot shed new light on the complexity of the satisfiability problem of *Q2MLO* and its derivatives,that we consider. This is so because *Q2MLO* includes the first-order *MLO* and even for this logic the satisfiability problem is known to be non-elementary.

Lemma 3.4. *If $\varphi(t,s)$ is a first-order MLO formula (with possible additional monadic variables), then $(\exists s)^{\leq t+1}_{>t}\varphi(t,s)$ is equivalent to a formula of $Q2MLO_0$. A similar claim is true for the quantifier $(\exists s)^{<t}_{\geq t-1}$.*

Proof. $(\exists s)^{\leq t+1}_{>t}\varphi(t,s)$ is true for some t and s in a structure, if either $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ is true, or else if $s = t+1$ is the first point beyond t that satisfies $\varphi(t,s)$. We will express $(\exists s)^{\leq t+1}_{>t}\varphi(t,s)$ as a disjunction of $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ with a formula that is true whenever $s = t+1$ is the first point beyond t that satisfies $\varphi(t,s)$. The following formula, which does not involve any metric quantifier, says that there is a first point s after t for which $\varphi(t,s)$ is true:

$$First_{\varphi}(t) \equiv \exists s(t < s \wedge \varphi(t,s) \wedge \forall v(t < v < s \rightarrow \neg\varphi(t,v)))$$

We look at the following statement:

$$First_{\varphi}(t) \wedge (\forall v)^{<t+1}_{>t}(\exists s)^{<v+1}_{>v}\varphi(t,s)$$

We claim that if $(\exists s)^{<t+1}_{>t}\varphi(t,s)$ is false then this statement is true if and only if $\varphi(t,t+1)$ is true. Indeed if $\varphi(t,t+1)$ is true then $First_{\varphi}(t)$ is true (as $s = t+1$ testifies) and for every v, $t < v < t+1$ this s testifies that $(\exists s)^{<v+1}_{>v}\varphi(t,s)$. On the other hand if the statement above is true then the point s that is the first past t with $\varphi(t,s)$ cannot be larger than $t+1$ or else any v which satisfies $t < v < t+1$ and $v \leq s-1$ cannot satisfy $(\exists s)^{<v+1}_{>v}\varphi(t,s)$). Therefore the first s with $\varphi(t,s)$ is necessarily $t+1$. Therefore

$$(\exists s)^{\leq t+1}_{>t}\varphi(t,s) \equiv (\exists s)^{<t+1}_{>t}\varphi(t,s) \vee (First_{\varphi}(t) \wedge (\forall v)^{<t+1}_{>t}(\exists s)^{<v+1}_{>v}\varphi(t,s))$$

However $(\forall v)^{<t+1}_{>t}(\exists s)^{<v+1}_{>v}\varphi(t,s))$ is not in $Q2MLO_0$ (or even in *Q2MLO*), because $(\exists s)^{<v+1}_{>v}\varphi(t,s))$ has the forbidden free variable t. The compositional method will supply a formula of $Q2MLO_0$ that is equivalent to $(\forall v)^{<t+1}_{>t}(\exists s)^{<v+1}_{>v}\varphi(t,s))$.

We recall (see Theorem 2.24) that there are formulas α_i and β_i such that *for every $t < v < s$*

$$M \models \varphi(t,s) \quad \text{iff} \quad M \models \bigvee_{i=1}^{q}(\alpha_i(t,v) \wedge \beta_i(v,s))$$

So that for every t

$$M \models (\forall v)^{<t+1}_{>t}(\exists s)^{<v+1}_{>v}\varphi(t,s) \quad \text{iff} \quad M \models (\forall v)^{<t+1}_{>t}\bigvee_{i=1}^{q}[\alpha_i(t,v) \wedge (\exists s)^{<v+1}_{>v}\beta_i(v,s)]$$

And the formula on the right is in $Q2MLO_0$.

It is much easier to replace the lower inequality by a weak one since for example:

$$(\exists s)^{<t+1}_{\geq t}\varphi(t,s) \equiv (\exists s)^{<t+1}_{>t}\varphi(t,s) \vee \varphi(t,t) \qquad \square$$

As a consequence of Lemma 3.4 we obtain the following proposition:

Proposition 3.5. *If $\varphi(t,s)$ is in Q2MLO$_0$ then $(\exists s)_{>t}^{<t+1}\varphi(t,s)$ is equivalent to a formula of Q2MLO$_0$. A similar claim is true for the quantifier $(\exists s)_{\geq t-1}^{<t}$.*

Proof. We prove by induction on the total number of metric quantifiers in $\varphi(t,s)$. If there are none, then $(\exists s)_{>t}^{<t+1}\varphi(t,s)$ is equivalent to a formula of *Q2MLO*$_0$ by Lemma 3.4.

Otherwise we eliminate some innermost occurrence of a metric quantifier in φ: Assume $(\exists v)_{>w}^{<w+1}\theta(v,w)$ is a subformula of φ with θ in non-metric monadic logic. Note that w may be t or s, and that v,w are the only variables free in θ. Therefore, there is some formula $\psi(t,s,X)$ that has one more monadic predicate variable X and one metric quantifier fewer than φ such that φ is obtained from $\psi(t,s,X)$ by substituting $(\exists v)_{>w}^{<w+1}\theta(v,w)$ for $X(w)$.

By the inductive assumption, there is a *Q2MLO*$_0$ formula $\psi_1(t,X)$ which is equivalent to $(\exists s)_{>t}^{<t+1}\psi(t,s,X)$.

Therefore, the formula α obtained by replacing $X(w)$ by $(\exists v)_{>w}^{<w+1}\theta(v,w)$ in $\psi(t,s,X)$ is equivalent to $(\exists s)_{>t}^{<t+1}\varphi(t,s)$. It is clear that α is in *Q2MLO*$_0$. $\square$

Next we treat intervals other than $(t,t+1)$:

Lemma 3.6. *If $\varphi(t,s)$ is a first-order MLO formula, then for any natural $n > 0$, $(\exists s)_{>t+n}^{<t+n+1}\varphi(t,s)$ is equivalent to a formula with only metric quantifier $(\exists u)_{>v+m-1}^{<v+m}$ for $0 < m \leq n$. A similar claim is true for the quantifier $(\exists s)_{>t-n-1}^{<t-n}$, and for the analogue formulas with weak inequality replacing one or two of the strict inequalities.*

Proof. The key fact is:

$$(\exists s)_{>t+n}^{<t+n+1}\varphi(t,s) \equiv (\exists v)_{>t+n-1}^{<t+n}(\forall w)_{>v}^{<v+1}(\exists s)_{>w}^{<w+1}\varphi(t,s)$$

Indeed, if the right hand side is true and there is such a v, then $t+n$ qualifies as a w and we see that the left hand side is true. In the opposite direction: If the left hand side is true then $v = s-1$ will make the right hand side true.

(Note that if one of the inequalities in $(\exists s)_{>t+n}^{<t+n+1}\varphi(t,s)$ is replaced by a weak inequality sign, then the corresponding inequality sign in $(\exists s)_{>w}^{<w+1}\varphi(t,s)$ will have to be changed to weak inequality).

Thus we reduced the bounds of the quantifiers by one, but the resulting formula is not even in *Q2MLO*, because the metric quantifier $(\exists s)_{>w}^{<w+1}$ is applied to a formula $\varphi(t,s)$ with a free variable $t \notin \{w,s\}$. We invoke a slight generalization to Theorem 2.24 that assures that there are formulas $\alpha_i(t,v)$ $\beta_i(v,w)$ and $\gamma_i(w,s)$ such that for $t < v < w < s$

$$M \models \varphi(t,s) \quad \text{iff} \quad M \models \bigvee_{i=1}^{q}(\alpha_i(t,v) \wedge \beta_i(v,w) \wedge \gamma_i(w,s))$$

We want to express the fact that there is some v in the interval $(t+n-1,t+n)$ such that every w in $(v,v+1)$ satisfies $\beta_i(v,w)$ and $\gamma_i(w,s)$ with one of the indices i such that $\alpha_i(t,v)$ holds, and with some s in the interval $(w,w+1)$. The formula begins by associating with a candidate point v between $t+n-1$ and $t+n$ the conjunction of formulas $\alpha_i(t,v)$ that it satisfies. Then for every w in $(v,v+1)$ there has to be a point s in $(w,w+1)$ such that $\beta_i(v,w) \wedge \gamma_i(w,s)$ holds for some i for which also $\alpha_i(t,v)$ is true. Therefore

$$(\exists v)_{>t+n-1}^{<t+n}(\forall w)_{>v}^{<v+1}(\exists s)_{>w}^{<w+1}\varphi(t,s)$$

is equivalent to

$$(\exists v)_{>t+n-1}^{<t+n}\bigvee_{A\subseteq\{1,\ldots,q\}}\left(\left[\bigwedge_{j\in A}\alpha_j(t,v)\right] \wedge \left[(\forall w)_{>v}^{<v+1}\bigvee_{k\in A}(\beta_k(v,w) \wedge (\exists s)_{>w}^{<w+1}\gamma_k(w,s))\right]\right) \qquad \square$$

Lemma 3.6 takes care of the inductive step in the proof of the following proposition:

Proposition 3.7. *If $\varphi(t,s)$ is in Q2MLO$_0$, then for any natural n, $(\exists s)_{>t+n}^{<t+n+1}\varphi(t,s)$ is equivalent to a formula in Q2MLO$_0$. A similar claim is true for the quantifier $(\exists s)_{>t-n-1}^{<t-n}$, and for the analogue formulas with weak inequality replacing one or two of the strict inequalities.*

Proof. The proof proceeds by induction on n, and for every n by the number of metric quantifiers of the form $(\exists s)_{>t+n}^{<t+n+1}$. The proof is similar to that of Proposition 3.5. $\square$

The last proposition almost completes the proof of the desired theorem:

Theorem 3.8 (*From Q2MLO to Q2MLO$_0$*). *Q2MLO$_0$ is as expressive as Q2MLO : Allowing only quantifications of the form $(\exists s)_{>t}^{<t+1}$ and $(\exists s)_{>t-1}^{<t}$ in the definition of Q2MLO does not restrict the expressive power of the logic.*

Proof

$$(\exists s)^{<t+m}_{>t+n}\varphi(t,s) \equiv (\exists s)^{\leq t+n+1}_{>t+n}\varphi(t,s) \vee (\exists s)^{\leq t+n+2}_{>t+1}\varphi(t,s) \vee \cdots \vee (\exists s)^{<t+m}_{>t+m-1}\varphi(t,s)$$

Therefore we can transform every *Q2MLO* formula into one that involves only intervals of length one. This formula can be transformed into a formula of $Q2MLO_0$ by repeatedly applying the last proposition inward out, starting with the metric quantifications which are the deepest. □

3.2. Reduction of Q2MLO to timer normal form

We want to show that every *Q2MLO* is effectively reducible to timer normal form, which will prove that *Q2MLO* is decidable. This will be done in two steps, which are of interest in themselves. First we will show that with fixed predicates N and E for natural numbers and for even numbers *Q2MLO* is reducible to *QMLO*. Then we will note that in *QMLO* there are formulas which state that the predicate N defines the natural numbers and E defines the even numbers. These results will be then put together to prove that *Q2MLO* is reducible to timer normal form.

We fix two of the predicates of *Q2MLO* and denote them by N and E. A structure over R^+ will be called *proper* if N is interpreted as the set of natural numbers and E as the set of even numbers.

Theorem 3.9 (*From Q2MLO to QMLO*)**.** *There is an algorithm which associates with every formula φ of Q2MLO a formula $\overline{\varphi}$ of QMLO, with the same first order free variables, such that the two formulas are equivalent in every proper structure.*

Proof. Given a formula φ we effectively transform it into a formula ϕ of $Q2MLO_0$, by theorem 3.8. If ϕ has no metric quantifier over a formula with two free variables then we are done. Else we will show how to reduce, one at a time, the number of such metric quantifiers. We choose some innermost metric quantifier that has two free variables in its range, and the subformula ψ that starts with it. Let us assume that it is a future quantifier (a past quantifier would have been treated similarly). Then $\psi = (\exists s)^{<t+1}_{>t}\theta(t,s)$ and θ is without metric quantifiers, and t,s are its only free variables.

We will show how to transform ψ into a formula of *QMLO* which can then be substituted instead of ψ, which will reduce the number of metric quantifiers.

Assume that a pair t and s in a proper structure satisfies $\theta(t,s) \wedge (t < s < t+1)$. Then either $t < s < \ulcorner t \urcorner$, where $\ulcorner t \urcorner$ is the smallest integer larger than t, or $t < s$ and there is a unique integer between them. The latter case is characterized by the fact that $\ulcorner t \urcorner$ is even and $\ulcorner s \urcorner$ is odd, or vice versa. Moreover, in this case the integer n between t and s satisfies: $\ulcorner t \urcorner = n = \llcorner s \lrcorner$, where $\llcorner s \lrcorner$ is the integral part of s. By the Composition Theorem 2.24 there are pairs $< \alpha_1(x,z),\beta_1(z,y) >, \cdots, < \alpha_q(x,z),\beta_q(z,y) >$ such that

$$M \models \psi(t,s) \quad \text{iff} \quad M \models \bigvee_{i=1}^{q}(\alpha(t,n) \wedge \beta_i(n,s))$$

It remains to show that this analysis can be expressed by a *QMLO* formula. To make the formulas more readable we introduce the notation: $[\llcorner t \lrcorner = v]$ for the *MLO* formula that states that v is the integral part of t:

$$[\llcorner t \lrcorner = v] \equiv (N(t) \wedge t = v) \vee (\neg N(t) \wedge N(v) \wedge (v < t) \wedge \forall s(v < s < t \rightarrow \neg N(s)))$$

The dual definition is slightly different; we want $\ulcorner t \urcorner$ to be $t+1$ if t is an integer, so that $\ulcorner t \urcorner - \llcorner t \lrcorner$ is always one unit long.

$$[\ulcorner t \urcorner = v] \equiv (t < v) \wedge N(v) \wedge \forall s(t < s < v \rightarrow \neg N(s))$$

We also define the predicate "t is an odd integer":

$$O(t) \equiv N(t) \wedge \neg E(t)$$

With this notation the properties that are equivalent to $(\exists s)^{<t+1}_{>t}\theta(t,s)$ can be written:

$$\exists v\{[\ulcorner t \urcorner = v] \wedge$$

$$\{\exists s((t < s < v) \wedge \theta(t,s)) \vee$$

$$\left(E(v) \wedge \bigvee_{i=1}^{q}\Big(\alpha_i(t,v) \wedge (\exists s)^{<t+1}_{>t}\exists w([\llcorner s \lrcorner = w] \wedge E(w) \wedge \beta_i(w,s))\Big)\right) \vee$$

$$\left(O(v) \wedge \bigvee_{i=1}^{q}\Big(\alpha_i(t,v) \wedge (\exists s)^{<t+1}_{>t}\exists w([\llcorner s \lrcorner = w] \wedge O(w) \wedge \beta_i(w,s))\Big)\right)\}\}$$

The second line accounts for the case where there is a solution s between t and $\ulcorner t \urcorner$, the third line accounts for the case where there is a solution s with $t < \ulcorner t \urcorner \leq s$, and $\ulcorner t \urcorner$ is even. The fact that $\llcorner s \lrcorner$ is also even assures that $\ulcorner t \urcorner \leq s < t+1$ and not $t < s < \ulcorner t \urcorner$. The fourth line accounts for the case where there is a solution s with $t < \ulcorner t \urcorner \leq s$, and $\ulcorner t \urcorner$ is odd. This covers all the possibilities and it is a formula of *QMLO*. □

Proposition 3.10. *There are formulas $Nat(X)$ and $Even(X,Y)$ in QMLO which define (in the model of the non-negative reals) the natural numbers and the even numbers, respectively. That is $Nat(X)$ holds in a structure iff X is the set of natural numbers in the structure, and $Even(X,Y)$ holds if X is the set of natural numbers and Y is the set of even numbers.*

Proof. Consider the following *QMLO* formula:

$$\forall t\{(\neg\exists s(s < t) \to X(t)) \wedge (X(t) \to (\forall s)_{>t}^{<t+1} \neg X(s)) \wedge (X(t) \to (\exists s)_{>t}^{\le t+1} X(s))\}$$

If the formula is satisfied by X then by the first conjunct X holds at zero. By the second conjunct, if X holds at t then X does not hold anywhere in the interval $(t,t+1)$. Therefore by the third conjunct X must hold on $t+1$. Therefore X is satisfied exactly by the natural numbers, and we can take this as the formula $Nat(X)$.

Next we use the notation $[\ulcorner t \urcorner = v]$ that we introduced and we look at the following *QMLO* formula:

$$Nat(X) \wedge \forall t\{(\neg\exists s(s < t) \to Y(t)) \wedge (Y(t) \leftrightarrow (X(t) \wedge \exists v([\ulcorner t \urcorner = v] \wedge \neg Y(v)))\}$$

Y holds at zero, it holds only for natural numbers, and it holds at a natural number if and only if it fails at the next one. Clearly this formula can be taken as $Even(X,Y)$. $\square$

The last two theorems together with previous results give us now the main technical result:

Theorem 3.11 (*From Q2MLO to Timer Normal Form*)**.** *There is an algorithm that associates with every Q2MLO formula a formula in timer normal which is equivalent over R^+ to the given formula.*

Proof. Let φ be a formula of *Q2MLO*. By Theorem 3.9 we can find a formula $\varphi'(N,E)$ of *QMLO* which is equivalent to φ whenever N and E are interpreted as the natural numbers and as the even numbers, respectively. The last proposition assures that every structure over R^+ has exactly one pair of sets that satisfy $Even(X,Y)$ namely the sets $X = N$ of naturals and the set $Y = E$ of even numbers. Therefore over R^+ the formula φ is equivalent to the formula $(\exists X,Y)[Even(X,Y) \wedge \varphi'(X,Y)]$. Since $\varphi'(X,Y)$ is in *QMLO* it is equivalent to a formula $\overline{\varphi'}(X,Y)$ in timer normal form. Similarly the formula $Even(X,Y)$ of *QMLO* is equivalent to a formula $\overline{Even}(X,Y)$ in timer normal form. Therefore φ is equivalent over R^+ to $(\exists X\exists Y)[\overline{Even}(X,Y) \wedge \overline{\varphi'}(X,Y)]$. Since the class of formulas in timer normal form is closed under conjunctions and the prefixing of existential set quantifiers we conclude that the last formula is indeed in timer normal form. $\square$

To state the next equivalence result concerning expressiveness we use the following notation:
For any logic T we denote by $T_\exists$ set of formulas of the form $\exists\overline{X}\varphi$ for all $\varphi \in T$.

Theorem 3.12. *The following logics are equivalent over R^+ (also over Q^+):*

(1) *$QMLO_\exists$*
(2) *$Q2MLO_\exists$*
(3) *Formulas in first order timer normal form.*

Proof. Every formula which is in first order timer normal form is trivially in $QMLO_\exists$ and Theorem 2.12 shows the other direction. Every formula of $QMLO_\exists$ is trivially in $Q2MLO_\exists$ and Theorem 3.11 shows that the converse is also true. $\square$

3.3. Decidable logics

Since Timer normal form formulas are decidable for satisfiability we obtain the following decidability theorem:

Theorem 3.13 (*Decidable logics*)**.**

(1) *Satisfiability and validity over R^+ are decidable for Q2MLO with respect to the class of all structures, or with respect to the class of structures with finite variability.*
(2) *Satisfiability over R^+ is decidable for $Q2MLO_\exists$, either with respect to the class of all structures, or with respect to the class of structures with finite variability.*
(3) *Satisfiability and validity over R^+ are decidable for every temporal logic whose modalities have truth tables in Q2MLO, either with respect to the class of all structures, or with respect to the class of structures with finite variability.*

Proof

(1) Let φ be in *Q2MLO*. By Theorem 3.11, we can compute a formula in first-order timer normal form which is equivalent to φ and check its satisfiability by Theorem 2.15. Applying the same test to $\neg\varphi$ which is also in *Q2MLO* tests if φ is valid or not.

(2) Let ψ be a formula in *Q2MLO*$_\exists$, i.e., ψ has the form $\exists\bar{W}\varphi$, with φ in *Q2MLO*. Then ψ is satisfiable iff φ is satisfiable. Hence, by (2), the satisfiability problem for *Q2MLO*$_\exists$ is decidable.
(3) Assume that every modality of a temporal logic has a truth table in *Q2MLO*. Then for every formula φ of this logic one can compute a *Q2MLO* formula $\psi(t)$ which is equivalent to φ. Hence, by (1) we obtain that this temporal logic is decidable. □

4. Generalized truth tables

A truth table $\varphi(t,Y_1,\ldots,Y_k)$ defines in every structure a function from k-tuples of subsets, which associates with the tuple $Y_1,\ldots,Y_k$ of subsets of a structure M the set of elements t in M that satisfy $\varphi(t,Y_1,\ldots,Y_k)$ in M. This is a special case of a more general way to define a function on all the structures in a given class of structures. Here is the formal notion of a "definable functional" in a class of structures.

Definition 4.1

(1) Let L be a language, and let $\mathcal{M}$ be a class of structures compatible with this language. Let $\varphi(X,Y_1,\ldots,Y_k)$ be a formula in L with no first order variables, and with no set variables except for those specified. φ is an *implicit definition of the functional* $X = f_\varphi^M(Y_1,\ldots,Y_k)$ if for every structure M in the class, and for any k subsets $Y_1,\ldots,Y_k$ of M, X is the only subset of M for which $M \models \varphi(X,Y_1,\ldots,Y_k)$.
(2) A modality $\mathcal{O}(Y_1,\ldots,Y_k)$ of a temporal logic has a *generalized truth table* $\varphi(X,Y_1,\ldots,Y_k)$ in the logic L above (over the class $\mathcal{M}$) if φ implicitly defines the operation of the modality; i.e., given subsets $Y_1,\ldots,Y_k$ of a structure M in $\mathcal{M}$,

$$M,t \models \mathcal{O}(Y_1,\ldots,Y_k) \mathit{iff}\, t \in f_\varphi^M(Y_1,\ldots,Y_k)$$

(3) More generally, $\varphi(X,Y_1,\ldots,Y_k)$ is a generalized truth table for the formula $\psi(Y_1,\ldots,Y_k)$ of the temporal logic if given subsets $Y_1,\ldots,Y_k$ of a structure M in $\mathcal{M}$,

$$M,t \models \psi(Y_1,\ldots,Y_k) \mathit{iff}\, t \in f_\varphi^M(Y_1,\ldots,Y_k)$$

If the logic is a second order logic, then this definition is a special case of the classical definition of a function defined by a formula. If the logic is first order then $X,Y_1,\ldots,Y_k$ are $k+1$ of its unary predicates, and the class is a class of structures in the signature from which $X,Y_1,\ldots,Y_k$ are excluded.

Note that if $\theta(t,Y_1,\ldots,Y_k)$ is a truth table for the modality $\mathcal{O}(Y_1,\ldots,Y_k)$ then the following formula in the variables $X,Y_1,\ldots,Y_k$ is a generalized truth table for the modality:

$$\forall t[X(t) \leftrightarrow \theta(t,Y_1,\ldots,Y_k)]$$

Therefore the notion of generalized truth table is more general than that of truth table. The notion is usually strictly more general than that of truth table. Thus it is known that there is no first-order *MLO* formula $\varphi(t)$ which defines over the naturals the set of even numbers or the set of odd numbers. I.e. there is no *MLO* formula $\varphi(t)$ such that $N \models \varphi(k)$ iff k is even. However, it is easy to write a first-order *MLO* formula $\psi(X)$ such that the only set which satisfies it over the naturals is the set of (say) odd numbers:

$$X(t) \longleftrightarrow \exists s(\neg X(s) \wedge \forall v(v \leq s \vee t \leq v))$$

0 is not in X since it has no predecessor, and therefore every odd number is in X and every even number is not. Therefore the only set X that satisfies the formula is the set of odd numbers.

It is also easy to extract from the discussion of the predicates describing the natural and the even numbers, a proof that the modality that holds at a point if and only if the point is in N, has a generalized truth table in *QMLO* . Yet it can also be shown that this modality has no truth table in *QMLO*.

For the particular logic *Q2MLO*$_\exists$ every modality that has a generalized truth table has also a truth table. We state it as a proposition:

Proposition 4.2. *Let $\varphi(X,Y_1,\ldots,Y_k)$ be a generalized truth table in Q2MLO$_\exists$ for a modality $\mathcal{O}$. That is,*

$$M,t \models \mathcal{O}(Y_1,\ldots,Y_k)$$

if and only if $t \in X$, where X is the unique set that satisfies $\varphi(X,Y_1,\ldots,Y_k)$. Then $\mathcal{O}$ has also a truth table (*not a generalized one*) *$\overline{\varphi}(t,Y_1,\ldots,Y_k)$ in Q2MLO$_\exists$, such that*

$$M,t \models \mathcal{O}(Y_1,\ldots,Y_k) \quad \text{iff} \quad M \models \overline{\varphi}(t,Y_1,\ldots,Y_k$$

Proof. Choose $\overline{\varphi}(t,Y_1,\ldots,Y_k)$:

$$\overline{\varphi}(t,Y_1,\ldots,Y_k) \quad \text{iff} \quad \exists X(\varphi(X,Y_1,\ldots,Y_k) \wedge X(t)).$$

Since $Q2MLO_\exists$ is closed under prefixing second order existential quantifiers, the formula is in $Q2MLO_\exists$. □

The following proposition extends this to general formulas in the modal logics which modalities defined by generalized truth tables, so that the decidability results for *Q2MLO* remain true.

Proposition 4.3

(1) *Assume that every modality of a temporal logic has a generalized truth table in $Q2MLO_\exists$. Then, every formula of this logic has a generalized truth table in $Q2MLO_\exists$.*
(2) *Satisfiability and validity over R^+ are decidable for every temporal logic whose modalities have (generalized) truth tables in $Q2MLO_\exists$, either with respect to the class of all structures, or with respect to the class of structures with finite variability.*
(3) *In particular, satisfiability and validity over R^+ are decidable for every temporal logic whose modalities have generalized truth tables in Q2MLO, either with respect to the class of all structures, or with respect to the class of structures with finite variability.*

Proof

(1) We proceed by the structural induction.
The case of atomic formulas is trivial.
For Boolean connectives. Assume that a formula $\psi(Y_1,\ldots,Y_k)$ has a generalized truth table $\varphi(X,Y_1,\ldots,Y_k)$ in $Q2MLO_\exists$. Therefore for every k sets $Y_1,\ldots,Y_k$ in a structure in $\mathcal{M}$ there is a unique subset X that satisfies $\varphi(X,Y_1,\ldots,Y_k)$. In particular the one and only witness to the formula $\exists X\varphi(X,Y_1,\ldots,Y_k)$ is this set. Hence, $\neg\psi$ has a generalized truth table $\alpha(X,Y_1,\ldots,Y_k)$ defined as

$$\exists Z\Big(\varphi(Z,Y_1,\ldots,Y_k)\wedge\forall tX(t)\leftrightarrow\neg Z(t)\Big).$$

This formula becomes an $Q2MLO_\exists$ formula in according to the strict definition after the second order existential quantifiers at the head of φ are pulled out in front of the parentheses.
The case of conjunction is treated similarly.
Finally, assume that a modality $\mathcal{O}(Y_1,\ldots,Y_k)$ has $Q2MLO_\exists$ generalized truth table $\varphi(X,Y_1,\ldots,Y_k)$, and formulas ψ_i have generalized truth table $\varphi_i(Y_i,\bar{Z}_i)$ in $Q2MLO_\exists$. Then, the formula $\mathcal{O}(\psi_1,\ldots,\psi_k)$ has generalized truth table

$$\exists Y_1\ldots Y_k\Big(\varphi\wedge\varphi_1\wedge\ldots\wedge\varphi_k\Big)$$

The last formula can be easily transformed into an equivalent $Q2MLO_\exists$ formula using equivalence

$$(\exists\bar{W}\alpha)\wedge(\exists\bar{V}\beta)\equiv\exists\bar{W}\bar{V}\Big(\alpha\wedge\beta\Big),\text{ provided}$$

$\bar{W}$ do not occur free in β and $\bar{V}$ do not occur free in α.
(2) It follows from Theorem 3.13(2) and from the observation that the construction in (2) was computable, that satisfiability is decidable for the logic. Since temporal logics are closed under negation, and since satisfiability for $\neg\varphi$ is the same as validity for φ, validity is also decidable for the logic.
(3) This is a special case of the previous observation, since *Q2MLO* is a sublogic of $Q2MLO_\exists$. It was explicitly declared because having a general truth table in *Q2MLO* does imply having a truth table in *Q2MLO*. □

5. Examples of stronger modalities

The Pnueli modalities are the most natural example of strong metric modalities that can be added to temporal logic, with truth tables in $Q2MLO_\exists$. Following this example we give some more examples, which may or may not be interesting for their own sake, but they are different in nature from the Pnueli modalities and they exemplify the possibilities. *Remember that each collection of these modalities, as well of any other collection of modalities that are defined in $Q2MLO_\exists$, will automatically yield a decidable temporal logic*. This is a major result of the paper.

5.1. The hierarchy of Pnueli modalities

We return to the Pnueli modalities (see Section 2.7), and the question of their decidability. We recall that for every n the n place modality $P_n(X_1,\ldots,X_n)$ says that there is an increasing sequence $t_0<\cdots<t_n$ in the coming unit interval of time such that for $i=1,\ldots,n$, $X_i(t_i)$ holds. Let $QTL(n)$ be the temporal logic $TL(\mathbf{U},\mathbf{S},P_1,\ldots,P_n)$.

Theorem 5.1 (*Hierarchy of temporal logics*)**.**

(1) *For every n the logic QTL(n) is decidable.*
(2) *These logics form a strict hierarchy; for every n the logic QTL(n + 1) is strictly more expressible than QTL(n).*

Proof. The modality $P_n(X_1,\ldots,X_n)$ has the truth table

$$(\exists t_n)^{<t_0+1}_{>t_0}(\exists t_1 \ldots \exists t_{n-1}([t_0 < t_1 < \cdots < t_n] \wedge [X_1(t_1) \wedge \cdots \wedge X_n(t_n)])$$

and the fact that this is a strict hierarchy was proven in [16]. □

5.2. Other modalities with truth tables in $Q2MLO_\exists$

In the process of proving the main theorem we defined in $Q2MLO_\exists$ the set of natural numbers and the set of even numbers (see Proposition 3.10). These definitions can be made into examples of truth tables for modalities that use explicitly the property of being a natural number, or that serve to count the elements that satisfy a formula modulo some natural number (whenever these elements form a sequence).

Examples: The following modalities $\lfloor\rfloor\varphi$, $\lfloor\Box\rfloor\varphi$, $\lfloor\Diamond_\infty\rfloor\varphi$ have truth tables in $Q2MLO_\exists$:

(1) $\lfloor\rfloor\varphi$ holds at a point t if and only if its integer value $\lfloor t\rfloor$ satisfies φ:
(2) $\lfloor\Box\rfloor\varphi$ which is true at t if and only if φ is true at every point between t and the next integer.
(3) $\lfloor\Diamond_\infty\rfloor\varphi$ is true at t if and only if φ is true at infinitely many points between t and the next integer.

For each modality $\mathcal{O}$ of the three we must find a formula $\Phi(X,Y)$ of $Q2MLO_\exists$ such that if $\Phi(A,B)$ holds then

$$M,A,B \models \Phi(X,Y) \quad \text{iff} \quad A = \{t \mid M,B,t \models \mathcal{O}Y\}$$

Here are the three formulas:

(1) $\Phi_1(X,Y)$ is the formula

$$\exists Z(Nat(Z) \wedge \forall t(X(t) \longleftrightarrow \exists s[s \le t \wedge Y(s) \wedge Z(s) \wedge \forall v(s < v \le t) \to \neg Z(v)])$$

(2) $\Phi_2(X,Y)$ is the formula

$$\exists Z(Nat(Z) \wedge \forall t(X(t) \longleftrightarrow \exists s[t < s \wedge Z(s) \wedge \forall v(t < v < s) \to Y(v)])$$

(We do not have to insist that s is actually the integer next to t. Any larger integer is just as good.)

(3) To improve readability we define the first order, monadic, non-metric, formulas $L^-(X,v)$ and $L^+(X,v)$ saying that v is a left accumulation point of the set X (or right accumulation point, respectively).

$$L^-(X,v) \equiv \forall w(v < w \to \exists u(v < u < w \wedge X(u)))$$

$\Phi_3(X,Y)$ is the formula

$$\exists Z(Nat(Z) \wedge \forall t(X(t) \longleftrightarrow \exists s[t < s \wedge Z(s) \wedge \forall v(t < v < s \to \neg Z(v)) \wedge$$

$$(L^-(Y,t) \vee L^+(Y,s) \vee \exists w[(t < w < s) \wedge (L^-(Y,w) \vee L^+(Y,w))])]))$$

The following example is a modality that differs in nature from the last examples. It shows that parity can be defined in a more general setting then just $Nat(X)$ and $Even(X,Y)$. It is definable in $Q2MLO_\exists$ by a modification of $Even(X,Y)$:

Example: The modality $E\varphi$ which holds at a point t if and only if t satisfies φ and the number of smaller elements that satisfy φ is finite and odd (so that t is at an even place when we enumerate the points satisfying φ).

6. Undecidability of more general modalities

The main feature of the logic $Q2MLO$ was the use of quantifiers of the form $(\exists s)^{<t+1}_{>t}\varphi$, provided at most t and s are free in φ. Is it possible to allow a more liberal reference to the interval $(t,t+1)$ and to produce a stronger logic with decidable satisfiability problem?

One way to do it is to allow other free variables in the formula φ. Another way is to allow a more complex reference to the interval. There is no systematic research of these questions, but every interesting modification that we attempted ended in an undecidable logic. The most obvious attempt is to allow a third free variable in the formula φ which has the metric quantifier in front.

Proposition 6.1. *Let T be the logic obtained from MLO by allowing the prefixing $(\exists s)^{<t+1}_{>t}$ to a formula $\varphi(v,s)$, where v may be different from t. Then the satisfiability problem is undecidable for T. Hence, if T' is the logic obtained from MLO by allowing three free variables to appear inside metric quantifiers, then T' is undecidable.*

Proof

It is well known that if the relation $u = t + 1$ is defined by a formula $\psi(t,u)$ in the logic then the logic is undecidable [2]. We define the formula:

$\psi(t,u) = (\forall s)^{<t+1}_{>t}(\exists v)^{<s+1}_{>s}(v = u)$

This formula is in the logic. Clearly $u = t + 1$ satisfies this formula. On the other hand if $u < t + 1$ then any s such that $u < s < t + 1$ will testify that $\psi(t,u)$ does not hold. And if $t + 1 < u$ then any s such that $t < s < t + 1$ for which $s \leq u - 1$ testifies that $\psi(t,u)$ does not hold. □

7. Extensions to other compositional logics

Before we conclude we would like to draw the readers's attention to the fact that the proofs of the results presented here were based entirely on composition theorems, and simple general properties of languages, models and interpretations. It follows that the same proofs can be applied in a more general context, for logics that have a composition property. We describe this direction, and give some particular examples.

Notation: Let L be a logic.

(1) We denote by *Q2L* the minimal extension of L defined as follows:

a) Every formula of L is in *Q2L*.
b) *Q2L* is closed under Boolean connectors and first order quantifications
and
c) *Q2L* is closed under applications of the "metric quantifiers":

If $\varphi(t_0,t)$ is a formula in *Q2L* with t and t_0 its only free first-order variable and $m < n$ are integers then $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,t)$ is a formula of *Q2L* (in the free variable t_0).

(2) The sublogic $Q2L_0$ of *Q2L* is defined as follows: A formula of *Q2L* is in $Q2L_0$ if all its metric quantifiers are of the form: $(\exists t)^{<t_0+1}_{>t_0}$ or $(\exists t)^{<t_0}_{>t_0-1}$.

Here are some examples of logics that have a composition theorem:

(1) **The second order monadic logic of order,** *SMLO*.
(2) **The weak monadic logic of order (weak** *SMLO***).** It has the same syntax as *SMLO* with the second-order quantifiers ranging over finite subsets.
(3) *SMLO* **with finite variability quantifiers**. It has the same syntax as *SMLO*, but the second-order quantifiers range over one place predicates with finite variability (free predicate variables and predicate names range over arbitrary predicates).
(4) **The counting** *MLO*. This is the extension of first-order *MLO* by the following " counting" quantifier $\exists^{(k,m)}$:

$M \models (\exists^{(k,m)}t)\varphi$ iff the number of t which satisfies φ in M is equal k modulo m

The composition theorem holds for *SMLO*, weak *SMLO*, *SMLO* with finite variability quantifiers and for counting *MLO*. These logics can also express the formulas $\overline{Timer}_n$ of 2.13, which were the key to the metric elimination. The composition theorem and expressibility of $\overline{Timer}_n$. are the only properties of first-order *MLO* which were used in our proofs in Section 3. Therefore, we have the following theorem:

Theorem 7.1. *Let Σ be a signature for SMLO (i.e., the number of named unary predicates is predetermined). Let $\mathcal{C}$ be either the class of all models in Σ or the class of models with finite variability predicates interpreting the named predicates. Let L be any one of the following logics: SMLO, weak SMLO, SMLO with finite variability quantifiers, or counting MLO. Then*

Q2L **is expressively equivalent to** $Q2L_0$ *There is an algorithm which for every $\varphi \in Q2L$ computes an equivalent formula in $Q2L_0$.*

Reduction to Timer normal form *There is an algorithm that associates with every Q2L formula φ an equivalent formula ψ of the form $\exists\bar{W}\beta$, where β is in L and $\exists\bar{W}$ is shorthand for a second order existential quantifier applied to a vector of second order variables.*

Decidability *The validity and satisfiability problems for Q2L are recursive in the satisfiability problem for L.*

Note that the second order logic has undecidable satisfiability problem. However, finite variability *SMLO* is decidable in the class of finite variability models and in the class of all models [19]. Weak *SMLO* is decidable over the reals and formulas of counting *SMLO* can be effectively translated into equivalent formulas of weak *SMLO*. Hence, we conclude by the last theorem that if L is one of the following logics: finite variability *SMLO*, weak *SMLO*, or counting *SMLO*, then *Q2L* is decidable.

8. Conclusion

The paper is a major part in a continuous research that develops temporal logic, pure and metric, in connection with classical predicate logic, in a way that is natural and general in the spirit of logic: the same temporal logic applies to the real line, to the rational line, to intervals, and in general to any metric linear order. The same temporal logic applies to finite variability signals and to general signals (which were ignored by the previous approach). The models are the usual relational structures that occur in logic and in mathematics.

As a benefit we get to use the rich theory of classical logic, including the theory of composition by Ehrenfeucht, Feferman-Vaught, Shelah and others. This allowed us to show that the full theory as developed before can be done in a unified logical way, that applies to all models. *From the logical point of view, there is nothing special about models with finite variability*.

This new framework allowed us to prove in [16] that no finite temporal logic can express all of Pnueli's modalities, so that metric temporal logic is inherently infinite. This informal claim must be formalized, and the framework developed here allows such a formalization. It is not clear how could one state and proof such a non-finiteness result without the logical framework developed here.

References

[1] R. Alur, T. Feder, T.A. Henzinger, The benefits of relaxing punctuality, Journal of the Association for Computing Machinery 43 (1996) 116–114.
[2] R. Alur, T.A. Henzinger, Logics and models of real time: a survey, in: de Bakker et al. (Eds.), Real Time: Theory and Practice, LNCS, vol. 600, 1992, pp. 74–106.
[3] R. Alur, T.A. Henzinger, Real time logic: complexity and expressiveness, Information and Computation, 104 (1993) 35–77.
[4] J.P. Burges, Y. Gurevich, The decision problem for temporal logic, Notre Dame Journal of Formal Logic 26 (1985) 115–128.
[5] H. Barringer, R. Kuiper, A. Pnueli, A really abstract concurrent model and its temporal logic, in: Proceedings of the 13th Annual Symposium on Principles of Programming Languages, 1986, pp. 173–183.
[6] A. Ehrenfeucht, An application of games to the completeness problem for formalized theories, Fundamenta mathematicae 49 (1961) 29–141.
[7] S. Feferman, R.L. Vaught, The first-order properties of products of algebraic systems, Fundamenta Mathematicae 47 (1959) 57–103.
[8] D.M. Gabbay, A. Pnueli, S. Shelah, J. Stavi, On the temporal analysis of fairness, in: 7th ACM Symposium on Principles of Programming Languages, Las Vegas, 1980, pp. 163–173.
[9] Y. Gurevich, Monadic second-order theories, in: J. Barwise, S. Feferman (Eds.), Model-Theoretic Logics, Springer-Verlag, 1985, pp. 479–506.
[10] T.A. Henzinger, It's about time: real-time logics reviewed, in: Concur 98, Lecture Notes in Computer Science, vol. 1466, 1998.
[11] T.A. Henzinger, J.-F. Raskin, P.-Y. Schobbens, The regular real-time languages, in: K.G. Larsen, S. Skyum, G. Winskel (Eds.), ICALP, LNCS, vol. 1443. Proceedings of 25th International Colloquium, Automata, Languages and Programming, ICALP'98, Aalborg, Denmark, July 13–17, Springer, 1998, pp. 580–591.
[12] Y. Hirshfeld, A. Rabinovich, A framework for decidable metrical logics, in: Proceedings of 26th ICALP Colloquium, LNCS, vol. 1644, Springer-Verlag, 1999, pp. 422–432.
[13] Y. Hirshfeld, A. Rabinovich, Quantitative temporal logic, in: Computer Science Logic 1999, LNCS, vol. 1683, Springer-Verlag, 1999, pp. 172–187.
[14] Y. Hirshfeld, A. Rabinovich, Logics for Real Time: Decidability and Complexity, Fundamenta Informaticae 62 (1) (2004) 1–28.
[15] Y. Hirshfeld, A. Rabinovich, Timer formulas and decidable metric temporal logic, Information and Computation 198 (2) (2005) 148–178.
[16] Y. Hirshfeld, A. Rabinovich, Expressiveness of metric modalities for continuous time, Logical Methods in Computer Science 3 (1) (2007)
[17] H. Kamp, Tense Logic and the Theory of Linear Order, Ph.D. Thesis, University of California L.A., 1968.
[18] Z. Manna, A. Pnueli, Models for reactivity, Acta Informatica 30 (1993) 609–678.
[19] A. Rabinovich, On the decidability of continuous time specification formalisms, Journal of Logic and Computation 8 (5) (1998) 669–678.
[20] S. Shelah, The monadic theory of order, Annals of Mathematics 102 (1975) 349–419.
[21] W. Thomas, Ehrenfeucht games, the composition method, and the monadic theory of ordinal words, in: Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht, Lecture Notes in Computer Science, vol. 1261, Springer-Verlag, 1997, pp. 118–143.
[22] T. Wilke, Specifying time state sequences in powerful decidable logics and time automata, in: Formal Techniques in Real Time and Fault Tolerance Systems, LNCS, vol. 863, 1994, pp. 694–715.