An Expressive Temporal Logic for Real Time

Yoram Hirshfeld and Alexander Rabinovich

School of Computer Science
Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel 69978
{joram, rabinoa}@post.tau.ac.il

Abstract. We add to the standard temporal logic with the modalities "Until" and "Since", a sequence of "counting modalities": For each n the modality $C_n(X)$, which says that X will be true at least at n points in the next unit of time, and its past counterpart $\overleftarrow{C}_n$, which says that X has happened at least n times in the last unit of time. We prove that this temporal logic is as expressive as can be hoped for; all the modalities that can be expressed in a strong natural decidable predicate logic framework, are expressible in this temporal logic.

1 Introduction

Temporal Logic based on the two modalities "Since" and "Until" (TL) is a popular among computer scientists as the framework for reasoning about a system evolving in time. By Kamp's theorem [18] this logic has the same expressive power as the first order monadic logic of order.

The two logics are (expressive) equivalent whether the system evolves in discrete steps or in continuous time, but for continuous time both logics can not express properties like: "X will happen within 1 unit of time". A natural metric modality say "X will happen exactly after one unit of time." Unfortunately, the extension of TL by this modality is undecidable. Over the years different decidable extensions of TL were suggested. The logic which was most extensively discussed was $MITL$ [2,1,10]. Other logics are described in [4,19,24]. We find the language QTL (quantitative temporal logic) which is presented in [13,14,15] more natural and convenient, and we will use it in the discussion. QTL has the two modalities "Until" and "Since", and two more modalities: $\diamondsuit_1 X$ - X will be true sometime within the next unit of time, and $\overleftarrow{\diamondsuit}_1 X$ - X was true sometime in the last unit of time.

We call these metric extensions of the pure temporal logic *the simple metric temporal logics*. They all have the same expressive power, which indicates that they capture a natural fragment of what can be said about the systems. This does not mean that they express all that needs to be said, and it was left to be determined whether these language are as expressive as can be hoped for, and if not, what needs to be added. Two important questions were not answered:

1. Is this logic expressive enough to express all the important properties about evolving systems?
2. If not, which modalities should we add?

R. Královič and P. Urzyczyn (Eds.): MFCS 2006, LNCS 4162, pp. 492–504, 2006.

Apparently A. Pnueli was the first to ask these questions, when he conjectured that the simple metric logics cannot express the requirement that X and then Y will both happen in the coming unit of time [2,24].

In [16] we proved Pnueli's conjecture, and we showed a sequence of modalities of the type that Pnueli suggested, such that no finite set of modalities can express all of them. Specifically: For every natural n we defined the "Pnueli modality" $Pnu_n(X_1, \ldots, X_n)$, which states that there is an increasing sequence $t_1, \ldots, t_n$ of points in the unit interval ahead such that t_i satisfies X_i. We also defined the weaker "Counting modalities" $C_n(X)$ which states that X will be true at least at n points in the unit interval ahead. We proved in [16] that:

- QTL (or $MITL$) with the added modalities $Pnu_1, \ldots, Pnu_n$ can not express the modality C_{n+1}
- No temporal logic with finitely many modalities can express all the modalities $C_n(X)$ for all natural numbers n.

This answers the first of the two questions above as negatively as can be imagined. It seemed to bring an end to the hope to extend Kamp's result, to define a simple temporal logic that extends plain temporal logic and is equivalent to a strong metric predicate logic.

In [12,14] we tried to identify the metric predicate logic that is best suited to deal with systems that evolve in time. A logic that is as expressive as possible, and yet simple to have a decidable validity and satisfiability problem. This logic can then serve to define modalities that will produce temporal logics that are decidable.

We started with the predicate logic that has also the $+1$ function alongside the order relation and the unary predicate variables. This language is too strong, as the $+1$ function allows for the encoding of Turing computations, and is clearly undecidable. We first identified the fragment of the predicate logic that corresponds to the temporal logic *QTL* (and to *MITL*). This is the "Quantified Monadic Logic of Order", **QMLO** that has atomic formulas $t = s$, $t < s$ and $X(t)$, is closed under Boolean connectors and first order quantifications, and under the "**metric quantifiers**": *If $\varphi(t)$ is a formula in QMLO with t its only free variable then $(\exists t)^{<t_0+1}_{>t_0}\varphi(t)$ is a formula of QMLO (in the free variable t_0).*

"Metric quantifiers" are just notations: $(\exists t)^{<t_0+1}_{>t_0}\varphi$ is shorthand for $(\exists t)[(t_0 < t < t_0 + 1) \wedge \varphi]$. This is a restricted form to use $+1$ function.

To extend the expressive power we then modified *QMLO* into *Q2MLO*; *Q2MLO* has the same atomic formulas as *QMLO* and is closed under the Boolean connectives, and first-order quantifiers, however the rule for metric quantifiers is changed to:

If $\varphi(t_0, t)$ is a formula in Q2MLO with t and t_0 its only free variable then $(\exists t)^{<t_0+1}_{>t_0}\varphi(t_0, t)$ and $(\exists t)^{<t_0}_{>t_0-1}\varphi(t_0, t)$ are formulas of Q2MLO.

At first glance the difference between *QMLO* and *Q2MLO* may look small, but it is actually very big. In particular for every (non metric) first order property φ we can state: "φ holds on some short interval (of length less than 1) that starts here".

In [12] we proved that *Q2MLO* has a decidable validity and satisfaction problem. Simple attempts to further modify *Q2MLO* like non trivial properties of the whole interval ahead, or permitting an additional third first-order variable to be free inside metric quantifier would make the logic as expressive as *FOMLO* with the function $+1$ (hence undecidable).

We demonstrated that *Q2MLO* is expressive decidable predicate logic in which modalities can be defined with the assurance that the resulting temporal logic is decidable. Therefore the second question above becomes:

> Can we add a nice (necessarily infinite) family of modalities to the pure temporal logic $TL(Until, Since)$ and obtain a temporal logic that is complete for *Q2MLO*?

Here we give the surprising answer:

Theorem: The pure temporal logic $TL(Until, Since)$ together with all the counting modalities C_n and $\overleftarrow{C}_n$ is expressively equivalent to *Q2MLO*.

The paper is divided as follows: In section 2 we recall the definitions and the previous results concerning the continuous time logics. Section 3 recalls the compositional method, which is a main tool in the proof of the theorem. In section 4 we prove the main theorem. Section 5 states further results.

2 Monadic Logic and Quantitative Temporal Logic

2.1 MLO - Monadic Logic of Order

We start with the standard definitions.

The monadic predicate logic of order - MLO has in its vocabulary *individual* (first order) variables $t_0, t_1, \dots,$, monadic *predicate* names $X_0, X_1, \dots,$, and one binary relation $<$ (the order).

The first order predicate language over this vocabulary is called the (first order) **Monadic Logic of Order (*FOMLO*)**. Note that since we are interested only in the first order language, $X_0, X_1, \dots,$ is a chosen sequence of constant predicate names, and not of set variables.

In this work **a structure for** *FOMLO* is a tuple $M = \langle \mathbb{R}, <, P_1, \dots, P_n \rangle$, where $\mathbb{R}$ is real line, with and $P_1, \cdots, P_n$, are one-place predicates (sets) that correspond to the predicate names in the logic.

As is common we will use the assigned formal names to refer to objects in the meta discussion. Thus we will write: $M \models \varphi[\tau_1, \dots, \tau_k; P_1, \dots, P_m]$ where M is a structure, φ a formula, $\tau_1, \cdots, \tau_k$ elements of M and $X_1, \dots, X_m$ predicates in M, instead of the correct but tedious form:

$$M, \tau_1, \dots, \tau_k; P_1, \dots, P_m \models_{MLO} \varphi(t_1, \dots, t_k; X_1, \dots, X_m),$$

where $\tau_1, \dots, \tau_k$ and $P_1, \cdots, P_m$ are names in the metalanguage for elements and predicates in M.

Everything in this paper remains true if we consider the class of structures whose domain is the non negative part of the real line, with 0 as first element.

In surveying the background, we will mention two more classes of structures: The class of Rational time, whose structures have the rational numbers as their domain, and the *Finite Variability class*, which is the of structures over the real line, such that every unary predicate changes its value only finitely often in any bounded interval of time.

2.2 Temporal Logics

Temporal logics use logical constructs called "modalities" to create a language that is free from variables and quantifiers:

The syntax of the Temporal Logic $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)}, \ldots)$ has in its vocabulary *monadic predicate names* $P_1, P_2, \ldots$ and a sequence of *modality names* with prescribed arity, $O_1^{(k_1)}, \ldots, O_n^{(k_n)}, \ldots$ (the arity notation is usually omitted). The formulas of this temporal logic are given by the grammar:

$$\varphi \ ::= True | \ P \mid \neg\varphi \mid \varphi \wedge \varphi \mid O^{(k)}(\varphi_1, \cdots, \varphi_k)$$

A temporal logic with a finite set of modalities is called a finite (base) temporal logic.

A structure for Temporal Logic, in this work, is the real line with monadic predicates $M = \langle \mathbb{R}, <, P_1, P_2, \ldots, P_n \rangle$, where the predicate P_i are those which are mentioned in the formulas of the logic. Every modality $O^{(k)}$ is interpreted in the structure M as an operator $O_M^{(k)} : [\mathbb{P}(A)]^k \to \mathbb{P}(A)$ which assigns "the set of points where $O^{(k)}[S_1, \ldots, S_k]$ holds" to the k-tuple $\langle S_1, \ldots, S_k \rangle \in \mathbb{P}(\mathbb{R})^k$. (Here $\mathbb{P}(\mathbb{R})$ denotes the set of all subsets of $\mathbb{R}$). Once every modality corresponds to an operator the semantics is defined by structural induction:

- for atomic formulas: $\langle M, t \rangle \models_{TL} P$ iff $t \in P$.
- for Boolean combinations the definition is the usual one.
- for $O^{(k)}(\varphi_1, \cdots, \varphi_k)$

  $$\langle M, t \rangle \models_{TL} O^{(k)}(\varphi_1, \cdots, \varphi_k) \quad \text{iff} \quad t \in O_M^{(k)}(A_{\varphi_1}, \cdots, A_{\varphi_k})$$

  where $A_\varphi = \{ \tau : \langle M, \tau \rangle \models_{TL} \varphi \}$ (we suppressed predicate parameters that may occur in the formulas).

For the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets A_{φ_i} of points satisfying φ_i and the set of points $O[A_{\varphi_1}, \ldots, A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

Truth Tables: A formula $\overline{O}(t_0, X_1, \ldots X_k)$ in the predicate logic L is a *Truth Table* for the modality $O^{(k)}$ if for every structure M

$$O_M(A_1, \ldots, A_k) = \{\tau \ : \ M \models_{MLO} \overline{O}[\tau, A_1, \ldots, A_k]\} \ .$$

The modalities *until* and *since* are most commonly used in temporal logic for computer science. They are defined through the following truth tables:

– The modality X **U** Y - "X *until* Y", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 < t_1 \wedge Y(t_1) \wedge \forall t(t_0 < t < t_1 \rightarrow X(t))).$$

– The modality X **S** Y - "X *since* Y", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 > t_1 \wedge Y(t_1) \wedge \forall t(t_1 < t < t_0 \rightarrow X(t))).$$

A central issue in this work is whether some temporal logic is equivalent to a fragment of predicate logic. We will define exactly what is meant by it:

Definition 1 (Expressive Equivalence). *Let L be a a fragment of predicate logic, and let TL be some temporal logic. Let $\mathcal{M}$ be a class of structures such that interprets both logics.*

1. *If for every formula $\varphi(t)$ of L with a single free variable there is a formula ϕ of TL, such that for every structure M in $\mathcal{M}$ and for every $t \in M$*

$$\langle M, t\rangle \models_{TL} \phi \quad \textit{iff} \quad M \models \varphi[t]$$

   then we say that TL is at least as expressive as L in the class $\mathcal{M}$.
2. *A similar condition says when L is at least as expressive as TL over $\mathcal{M}$.*
3. *If both conditions hold we say that TL and L are expressively equivalent over $\mathcal{M}$, or that they have the same expressive power.*

If the modalities of a temporal logic have truth tables in a predicate logic then the temporal logic is equivalent to a fragment of the predicate logic. Formally:

Proposition 1. *If every modality in the temporal logic TL has a truth table in the logic FOMLO then to every formula $\varphi(X_1, \ldots, X_n)$ of TL there corresponds effectively (and naturally) a formula $\overline{\varphi}(t_0, X_1, \ldots X_n)$ of FOMLO such that for every M, $\tau \in M$ and predicates $P_1, \ldots, P_n$*

$$\langle M, \tau, P_1, \ldots, P_n\rangle \models_{TL} \varphi \quad \text{iff} \quad \langle M, \tau, P_1, \ldots, P_n\rangle \models_{MLO} \overline{\varphi}\ .$$

In particular the temporal logic $TL($ **U** , **S** $)$ with the modalities "until" and "since" corresponds to a fragment of first-order MLO.

The two modalities **U** and **S** are also enough to express all the formulas of first-order $FOMLO$ with one free variable:

Theorem 2. *([18,9]) The temporal logic $TL($ **U** , **S** $)$ is expressively complete for FOMLO over the two canonical structures: For every formula of FOMLO with at most one free variable, there is a formula of $TL($ **U** , **S** $)$, such that the two formulas are equivalent to each other, over the positive integers (discrete time) and over the real line (continuous time).*

2.3 The Simple Metric Logics: Quantitative Temporal Logic, and Quantitative Monadic Logic of Order

The logics MLO and $TL($ **U** , **S** $)$ are not suitable to deal with statements like "X will occur within one unit of time". It might be tempting to include in the monadic logic of order also a unary function symbol to denote the function $t+1$. This however makes the language undecidable. Moreover, seemingly weak fragments of this logic are undecidable. The corresponding modification of the temporal logic would add to the logic the modality $S(X)$ that holds at a point t if $t+1 \in X$. This also results in an undecidable logic. For the last 20 years languages that can express such properties and are decidable were proposed and investigated ([4,19,8,24,10,13,15]), and most notorious, logic $MITL$ [2,1,10]. We will use as a framework the *Quantitative Temporal Logic, QTL* which was introduced in [12,13,14]. All these logics are expressively equivalent [13]. QTL is defined as follows:

Definition 3 (Quantitative Temporal Logic). *QTL, quantitative temporal logic is the logic $TL(\mathbf{U}, \mathbf{S})$ enhanced by the two modalities: $\Diamond_1 X$ and $\overleftarrow{\Diamond}_1 X$. These modalities are defined by the tables with free variable t_0:*

$$\Diamond_1 X : \qquad \exists t((t_0 < t < t_0 + 1) \land X(t)) \tag{3}$$

$$\overleftarrow{\Diamond}_1 X : \qquad \exists t((t-1 < t < t_0) \land X(t)) \ . \tag{4}$$

The temporal logic QTL is complete for a natural fragment of the monadic logic of order, enriched with the $+1$ function:

Definition 4 (Quantitative Monadic Logic of Order). *$QMLO$,* quantitative monadic logic of order *is the predicate logic that has atomic formulas $t = s$, $t < s$ and $X(t)$, where X ranges over unary predicate names; it is closed under Boolean connectors and first order quantifications, and has the following rule for the "***metric quantifiers***":*

> If $\varphi(t)$ is a formula in $QMLO$ with t its only free variable and $m < n$ are integers then $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t)$ is a formulas of $QMLO$.

Recall that the metric quantifier $(\exists t)^{<t_0+n}_{>t_0+m}\varphi$ is shorthand for $\exists t(t_0 + m < t < t_0 + n \land \varphi)$.

Theorem 5. *([13,15]) The temporal logic QTL is expressively equivalent to QMLO over the full, or positive half, real line.*

2.4 The Limited Expressive Power of the Simple Metric Logics

There was no reason to believe that the simple metric logics like QTL have comprehensive expressive power. A. Pnueli raised this question, and he conjectured that the modality $Pnu_2(X, Y)$ is not expressible in $MITL$, where $Pnu_2(X, Y)$ says that X and then Y will be true at points in the next unit of time [2,24].

In [16] we proved Pnueli's conjecture, and we strengthened it significantly. To do this we defined for every natural n the "Pnueli modality" $Pnu_n(X_1, \ldots, X_n)$, which states that there is an increasing sequence $t_1, \ldots, t_n$ of points in the unit interval ahead such that t_i satisfies X_i. We also defined the weaker "Counting modalities" $C_n(X)$ which states that X is true at least at n points in the unit interval ahead. I.e, $C_n(X) = Pnu_n(X, \ldots, X)$. With these modalities we proved:

Theorem 6. *1. QTL (or MITL) with the added modalities $Pnu_1, \ldots, Pnu_n$ can not express the modality C_{n+1}.*

2. No temporal logic with finitely many modalities can express all the modalities $C_n(X)$ for all natural numbers n. Hence no finite temporal logic will suffice to express everything of interest.

2.5 The Predicate Metric Logic *Q2MLO*

In classical monadic logic there is a natural logic suitable to deal with evolving systems, the logic *Q2MLO* , which was introduced in [12] .

Definition 7. *Q2MLO is the predicate logic that has atomic formulas $t = s$, $t < s$ and $X(t)$, where X ranges over unary predicate names; it is closed under Boolean connectors and first order quantifications, and has the following rule for the "***metric quantifiers***":*

> If $\varphi(t_0, t)$ is a formula in *Q2MLO* with free first-order variables in $\{t_0, t\}$ and $m < n$ are integers then $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0, t)$ is a formulas of *Q2MLO*.

In [12] it was shown that:

Theorem 8. *The validity and satisfiability problem is decidable for Q2MLO, over continuous time, whether we are interested in the class of models with finite variability, or in the class of all models.*

QMLO and *Q2MLO* are both decidable, and they look similar. But there are some major differences:

1. It is very easy to express in *Q2MLO* properties that are not expressible in *QMLO* . Thus for example Pnueli's modality $Pnu_n(X_1, \ldots, X_n)$ has *Q2MLO* truth table $(\exists t)^{<t_0+1}_{>t_0}(\exists t_1, \cdots, t_n)[(t_0 < t_1 < \cdots < t_n < t) \wedge (X_1(t_1) \wedge \cdots \wedge X_n(t_n))]$
2. *Q2MLO* seems strong enough to express all the decidable modalities that we found in the literature, and we have yet to see a natural formula in some decidable logic that cannot be expressed in *Q2MLO*.
3. *QMLO* is expressively equivalent to *QTL* that has four modalities only. *Q2MLO* is not equivalent to any temporal logic with a finite number of modalities.

3 Elements of Composition Method

Here we recall some elements of the compositional method. This method will play an important role in the proofs of Section 4.

The "compositional method" applies to the case where a structure is composed from simpler structures, and the theory of the composite structure can be reduced to the theory of its components. Ehrenfeucht used it in [6] and our proofs follow his work. The method was developed and used by Fefeman-Vaught [7], Shelah [22] and others (see e.g., surveys [11,23,20]). Here is an introduction to what we need:

The structures of the form $M = (A, <, P_1, \ldots, P_m)$, where $<$ is a linear order on a set A and $P_i \subseteq A$ are called m-labelled chains.

Two m-labelled chains M, M' are called k-equivalent (written: $M \equiv_k M'$) if $M \models \varphi \;\Leftrightarrow\; M' \models \varphi$ for every sentence φ of quantifier depth k. This is an equivalence relation between labelled chains; its equivalence classes are called k-types for the given signature with $<$ and m unary predicate symbols. Let us list some fundamental and well-known properties of k-types.

Proposition 9. *1. For every m and k there are only finitely many k-types of m-labelled chains.*

2. For each k-type τ there is a sentence (called "characteristic sentence") which defines τ (i.e., is satisfied by a labelled m-chain iff it belongs to τ). For given k and m a finite list of characteristic sentences for all the possible k-types can be computed. (We take the characteristic sentences as the canonical representations of k-types. Thus, for example, transforming a type into another type means to transform sentences.)

3. Each sentence φ is equivalent to a (finite) disjunction of characteristic sentences; moreover, this disjunction can be computed from φ.

Given m-labelled chains M_0, M_1 we write $M_0 + M_1$ for their concatenation (ordered sum); the domain of $M_0 + M_1$ is the union of the domains of M_0 and M_1 (we assume that these domains are disjoint), the interpretation of a unary predicates P is the union of its interpretation in M_0 and in M_1, all elements of M_0 are less than all elements of M_1 and if two elements are in M_i, then their order in $M_0 + M_1$ is the same as in M_i.

We need the following composition theorem for ordered sums:

Theorem 10 (Composition Theorem). *The k-types of m-labelled chains M_0, M_1 determine the k-type of the ordered sum $M_0 + M_1$:*

This theorem justifies the notation $\tau_1 + \tau_2$ for the k-type of an m-chain which is the sum of two m-chains of k-types τ_1 and τ_2.

Corollary 11. *For every formula $\varphi(s,t)$ with free first-order variables s and t there is a finite list of pairs of formulas $\tau_1(s,t), \sigma_1(s,t), \ldots, \tau_k(s,t), \sigma_k(s,t)$, such that for any labelled chain M and three points $p < q < r$:*

$$M \models \varphi(p,r) \textit{ iff for some } i \le k,\ M \models \tau_i(p,q) \textit{ and } M \models \sigma_i(q,r).$$

Moreover, the list $\tau_i(s,t), \sigma_i(s,t)$ is computable from φ.

4 Completeness of the Counting Modalities

Let TLC be the temporal logic $TL(\mathbf{U}, \mathbf{S})$ together with all the counting modalities $C_n(X)$ and $\overleftarrow{C}_n(X)$. We start to prove that TLC is equivalent to

Q2MLO. First we state two lemmas, that improve our understanding of the logic *Q2MLO*. The first lemma allows us to replace *Q2MLO* by a large temporal logic, whose modalities are all the modalities that can be defined by *Q2MLO* formulas with some restriction.

Lemma 12. *Let Γ be the set of all modalities with truth table of the form $(\exists t)^{<t_0+n}_{>t_0+m}\phi(t_0,t,X_1,\ldots,X_k)$ and $(\exists t)^{<t_0-m}_{>t_0-n}\phi(t_0,t,X_1,\ldots,X_k)$, with $0 \leq m < n$, and where ϕ is a formula of FOMLO with the free first-order variables in $\{t_0,t\}$. Then, TL(Γ, Until, Since) is expressively equivalent to Q2MLO.*

We omit the proof of Lemma 12 for lack of space; it proceeds by the induction of nesting of metric quantifiers. The second lemma shows that *Q2MLO* can be defined using only simple intervals of length 1 in the definition. It will be the first of three applications of the composition method.

Lemma 13. *For every Q2MLO formula ψ there exists an equivalent Q2MLO formula ϕ which uses only metric quantifiers of the form $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$.*

Proof. (Sketch) We exemplify the method proving the following two statement. Let $\varphi(t,s)$ be a pure *FOMLO* formula.

1. if $n>1$ then $(\exists s)^{<t_0+n}_{>t_0}\varphi(t_0,s)$ is equivalent to a formula which uses metric quantifiers $(\exists s)^{t+m}_{t}$ where $m<n$.
2. if $n>0$ then $(\exists s)^{<t_0+n+1}_{>t_0+n}\varphi(t_0,s)$ is equivalent to a formula which uses metric quantifiers $(\exists s)^{t+n}_{t+n-1}$ and $(\exists s)^{t+1}_{t}$.

All other cases can be treated similarly, or proven from these cases by induction on the numbers that appears in the metric quantifiers together with the technique used in the proof of Lemma 12.

(1) By Corollary 11 there are formulas $\tau_1,\sigma_1,\ldots,\tau_k,\sigma_k$, such that for any three points $p<q<r$, $\varphi(p,r)$ is true iff for some $i\leq k$, $\tau_i(p,q)$ and $\sigma_i(q,r)$ hold. It follows that

$$(\exists s)^{<t_0+n+1}_{>t_0}\varphi(t_0,s) \equiv \bigvee_{i=1}^{k}(\exists q)^{<t_0+n}_{>t_0}(\tau_i(t_0,q)\wedge(\exists s)^{<q+1}_{>q}\sigma_i(q,s))$$

Indeed if the right side of the equivalence is true then for the q and s that satisfy it there is some i such that $\tau_i(t_0,q)$ and $\sigma_i(q,s)$ and therefore $\varphi(t_0,s)$. The distance from t_0 to q is less than n and the further distance to s is less than 1. Hence s testifies the left side of the equivalence. If on the other hand s testifies the property on the left, then if $t_0+n<s$ we choose any q between $s-1$ and t_0+n. If $s\leq t_0+n$ then we may choose any q smaller than s which is larger than $s-1$ (and t_0). This will satisfy the right side.

(2) We deal similarly with $(\exists s)^{<t_0+n+1}_{>t_0+n}\varphi(t_0,s)$: By the corollary to the compositional theorem there are pairs of formulas: $\sigma_1,\tau_1,\ldots\sigma_k,\tau_k$, such that for every three points $t<p<s$ we have $\varphi(t,s)$ iff for some $i\leq k$, $\tau_i(t,p)$ and $\sigma_i(p,s)$. Furthermore for every $i<k$ there are m_i pairs of formulas $\pi_{i,1},\eta_{i,1},\ldots\pi_{i,m_i},\eta_{i,m_i}$

such that for every point q between p and s, $\sigma_i(p,s)$ holds iff for some $j \leq m_i$ both $\pi_{i,j}(p,q)$ and $\eta_{i,j}(q,s)$ hold. We will show that

$$(\exists s)^{<t_0+n+1}_{>t_0+n}\varphi(t_0,s) \equiv \bigvee_{i=1}^{k}(\exists p)^{<t_0+n}_{>t_0+n-1}[\tau_i(t_0,p) \wedge (\forall q)^{<p+1}_{>p}\bigvee_{j=1}^{m_i}(\pi_{i,j}(p,q) \wedge (\exists s)^{<q+1}_{>q}\eta_{i,j}(q,s))]$$

The expression on the right is already of the right form. Assume that there is some s that makes the left side true. Then we choose $p = s - 1$ and we will show that this p satisfies the right formula. Indeed for every q such that $p < q < p + 1 = s < q + 1$ the point s is the unit interval to the right of q, and since $\varphi(t_0, s)$ is true there is some $i < k$ for which $\tau_i(t_0, p)$ and $\sigma_i(p, s)$ hold. And since q lies between p and s, there is some $j < m_i$ for which $\pi_{i,j}(p, q)$ and $\eta_{i,j}(q, s)$ hold. Therefore the right statement is true. In the other direction, assume that the i-th disjunct of the right statement is true. Hence, there is some p in the interval $(t_0 + n - 1, t_0 + n)$ such that $\tau_i(t_0, p)$ is true and $q = t_0 + n$ is in the interval $(p, p + 1)$ and therefore there is s in the interval $(t_0 + n, t_0 + n + 1)$ and there is an index $j < m_i$ for which $\pi_{i,j}(p, q)$ and $\eta_{i,j}(q, s)$ is true. It follows from the main property of the formulas τ, π and η that $\varphi(t_0, s)$ is true. Since s is the interval $(t_0 + n, t_0 + n + 1)$, the left statement is true. □

Theorem 14 (Main theorem). *The temporal logic TLC and the monadic logic Q2MLO are expressively equivalent.*

Proof. (Sketch) In one direction, every formula of TLC is equivalent to a formula of *Q2MLO* as all its modalities have truth tables in *Q2MLO*.

For the other direction, by the last lemma, it suffices to prove that every *Q2MLO* formula ϕ with metric quantifiers of the form $(\exists s)^{<t+1}_{>t}$ and $(\exists s)^{<t}_{>t-1}$, can be expressed in TLC. To simplify the discussion, we avoid the careful yet cumbersome distinction between the free variables of a formula $\varphi(t, s)$ and their intended interpretation in the model. We will speak freely of "the interval (t, s)", and say that "the interval satisfies the formula φ", instead of "the interpretation satisfies $\varphi(t, s)$". We must show that $(\exists s)^{<t+1}_{>t}\varphi(t, s)$ can be expressed by a TLC formula, that holds at t. Following a technical inductive argument we may assume that φ is of pure monadic logic. We divide the proof into steps.

1. It maybe that $(\exists s)^{<t+1}_{>t}\varphi(t, s)$ holds because there is a sequence of points s to the right of t that converge to t, and such that $\varphi(t, s)$ holds. We call such t a *φ-limit*. This property can be written in pure "Until, Since" temporal logic. We may therefore look for a TLC formula that expresses $(\exists s)^{<t+1}_{>t}\varphi(t, s)$ under the assumption that t is not a φ-limit. We can then add the temporal formula that says that t is a *φ-limit*, as a disjunct.
2. An interval $[x, y]$ will be called a *φ-interval* if
   (a) either $\varphi(x, y)$ holds, or y is a limit of points z to the right, for which $\varphi(x, z)$ holds.
   (b) $[x, y]$ is minimal, in that no $x < z < y$ satisfies $\varphi(x, z)$.

   If $[x, y]$ is a φ-interval, then x will be called a *left φ point* and y is a *right φ point*. x is the *left φ-partner* of y, and y is the *right φ-partner* of x.

Every left φ point has a unique φ-interval and a unique right partner. (the dual does not hold for a right φ point).
Claim: Assume that $\varphi(x, y)$ holds, and x is not a φ-*limit*. Then:
(a) x is a left φ point, of some φ-interval.
(b) If $x < y < x + 1$ then $x < y' < x + 1$, where y' is the right φ partner for x .
(Just choose y' to be the greatest lower bound of the points y that satisfy $\varphi(x, y)$). Therefore from now on we replace $\varphi(t, s)$ in $(\exists s)^{\leq t+1}_{>t}\varphi(t, s)$ by ($[t, s]$ is a φ-interval). From now on when we say φ we mean this formula. It is not difficult to see that the properties : "x is a left φ point", "x is a right φ point" and "x, y is a φ pair", can be expressed in pure monadic logic of order.

3. The crucial property is that there is a simple computable bound on the length of a proper decreasing sequence of φ-intervals. We say that the interval $[x, y]$ is greater than $[x', y']$ if $x < x' < y' < y$. Let $\tau_1, \sigma_1, \ldots, \tau_k, \sigma_k$ as in the composition theorem 10, be the formulas such that for $x < z < y$, $\varphi(x, y)$ holds, iff one of the pairs $\sigma_i(x, z)$ and $\tau_i(z, y)$ hold. Then:
Claim: Every proper decreasing sequence of φ-intervals has depth at most k.
Proof: if $x_1 < \cdots < x_{k+1} < y_{k+1} < \cdots < y_1$ is such that $[x_i, y_i]$ is a φ interval and if z is in all these intervals then for every i there is some formula τ_j such that $[z, y_i]$ satisfies τ_j. By the pigeonhole principle there are some $y_i < y_{i'}$ such that both $[z, y_i]$ and $[z, y_{i'}]$ satisfy the same τ_j. Therefore both $[x_{i'}, y_i]$ and $[x_{i'}, y_{i'}]$ satisfy φ, which is impossible, as $[x_{i'}, y_{i'}]$ is a minimal such interval.
4. For every φ-interval $[t, s]$ we define its rank to be the length of the longest proper decreasing sequence of φ-intervals that starts with $[t, s]$. We denote by $\varphi_i(t, s)$ the claim that $[t, s]$ is a φ-interval of rank i, and by $r_i(s)$ the claim that s is the right endpoint of such an interval. Clearly:
(a) Every φ-interval has rank between 1 and k.
(b) The formulas $\varphi_i(x, y)$ and $r_i(s)$ are expressible in pure monadic logic of order.
5. We come to a third application of the composition theorem:
Claim: Let t be a point, and let i be some rank. Let $\tau_1, \sigma_1, \ldots, \tau_p, \sigma_p$ as in the composition theorem 10, be the formulas such that for $x < z < y$, $\varphi_i(x, y)$ holds, iff one of the pairs $\sigma_j(x, z)$ and $\tau_j(z, y)$ hold. Then only the first $p + 1$ points to the right of t that satisfy $r_i(s)$ can be right endpoints of a φ-interval $[u, s]$ of rank i, such that $u \leq t$.
Proof: Assume that $s_1 < s_2 < \cdots < s_l$ is a sequence of points satisfying $r_i(s)$, and $t_1, \cdots, t_l$ are the corresponding left hand partners. Then $t_1 < \cdots < t_l$, or else one interval would contain another, and their rank would not be the same. It follows that all the right hand points of intervals of rank i that correspond to partners up to t come before those that correspond to partners greater than t. Moreover, there are at most $p + 1$ of them, or else there would be $j \neq j'$ such that $[t, s_j]$ and $[t, s_{j'}]$ that satisfy the same τ_v, and $[t_j, t]$ and $[t_{j'}, t]$ satisfy σ_v, therefore $[t_j, s_j]$ and $[t_j, s_{j'}]$ are φ-intervals of rank i with the same left endpoint which is impossible.

6. We may now collect the pieces: $(\exists s)^{<t+1}_{>t}\varphi_i(t,s)$ holds iff one of the following disjuncts for $j = 1,\ldots,p+1$ holds:
   There are at least j points between t and $t+1$ that satisfy $r_i(s)$ and t together with the j^{th} point among the points larger than t that satisfies $r_i(s)$, satisfy also $\varphi_i(t,s)$.
   Note that everything is pure monadic logic (and hence pure temporal), except the claim of the existence of j points with a pure monadic property, between t and $t+1$. Therefore this can be expressed by an TLC formula. It remains now to take the disjunction of these formulas,over the ranks (and add one disjunct for the case that t is a φ-limit, as in item 1).

5 Conclusion and Further Results

We added to the temporal logic $TL(Until, Since)$ all the modalities $C_n(X)$ - "X will be true at least at n points in the next unit of time", and $\overleftarrow{C}_n(X)$ -"X was true at least at n points in the last unit of time". The resulting temporal logic is complete for a strong, yet decidable monadic logic of order, $Q2MLO$.

The expressive completeness result can be extended to the rational time line. However, $TL(Until, Since)$ is not expressively complete for the $FOMLO$ over the rational line. Stavi found two modalities $Until_{St}, Since_{St}$ which have the same expressive power over the class of all linear orders as the $FOMLO$ [8]. We can prove that the Stavi modalities together with $C_n(X)$ and $\overleftarrow{C}_n(X)$ have the same expressive power over the rationals as $Q2MLO$.

As with the pure temporal logic $TL(Until, Since)$ there is a gap between the complexity (and succinctness) of the temporal logic and that of the corresponding predicate logic. In [17] we proved that the satisfiability problem for the temporal logic $TL(Until, Since, \{C_n, \overleftarrow{C}_n\}_{n=1}^{\infty})$ is $PSPACE$ complete.

References

1. R. Alur, T. Feder, T.A. Henzinger. The Benefits of Relaxing Punctuality. Journal of the ACM 43 (1996) 116-146.
2. R. Alur, T.A. Henzinger. Logics and Models of Real Time: a survey. In Real Time: Theory and Practice. Editors de Bakker et al. LNCS 600 (1992) 74-106.
3. J. A. Brzozowski, R. Knast. The dot depth hierarchy of star free languages is infinite. J. of Computing Systems Science 16 (1978) 37-55.
4. Baringer H. Barringer, R. Kuiper, A. Pnueli. A really abstract concurrent model and its temporal logic. Proceedings of the 13th annual symposium on principles of programing languages (1986), 173-183.
5. H.D. Ebbinghaus, J. Flum, Finite Model Theory. Perspectives in mathematical logic, Springer (1991).
6. A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. Fundamenta Mathematicae 49 (1961), pp. 129-141.
7. S. Feferman and R.L. Vaught. The first-order properties of products of algebraic systems. *Fundamenta Mathematicae* **47**:57–103, 1959.

8. D.M. Gabbay, I. Hodkinson, M. Reynolds. Temporal Logics volume 1. Clarendon Press, Oxford (1994).
9. D.M. Gabbay, A. Pnueli, S. Shelah, J. Stavi. On the Temporal Analysis of Fairness. 7th ACM Symposium on Principles of Programming Languages. Las Vegas (1980) 163-173.
10. T.A. Henzinger It's about time: real-time logics reviewed. in Concur 98, Lecture Notes in Computer Science 1466, 1998.
11. Y. Gurevich (1985). Monadic second-order theories. In *Model-Theoretic Logics*, (J. Barwise and S. Feferman, eds.), 479-506, Springer-Verlag.
12. Y. Hirshfeld and A. Rabinovich, A Framework for Decidable Metrical Logics. In Proc. 26th ICALP Colloquium, LNCS vol.1644, pp. 422-432, Springer Verlag, 1999.
13. Y. Hirshfeld and A. Rabinovich. Quantitative Temporal Logic. In Computer Science Logic 1999, LNCS vol. 1683, pp 172-187, Springer Verlag 1999.
14. Y. Hirshfeld and A. Rabinovich, Logics for Real Time: Decidability and Complexity. Fundam. Inform. 62(1): 1-28 (2004).
15. Y. Hirshfeld and A. Rabinovich, Timer formulas and decidable metric temporal logic. Information and Computation Vol 198(2), pp. 148-178, 2005.
16. Y. Hirshfeld and A. Rabinovich, Expressiveness of Metric Modalities for Continuous Time. In Proccedings of International Computer Science Symposium in Russia, LNCS vol. 3967, pp. 211-220, Springer Verlag, 2006.
17. Y. Hirshfeld and A. Rabinovich, On The complexity of the temporal logic with counting modalities. Manuscript, 2006.
18. H. Kamp. Tense Logic and the Theory of Linear Order. Ph.D. thesis, University of California L.A. (1968).
19. Z. Manna, A. Pnueli. Models for reactivity. Acta informatica 30 (1993) 609-678.
20. Makowsky, J. A. 2004. Algorithmic aspects of the Feferman-Vaught theorem. *Annals of Pure and Applied Logic 126(1-3)*, 159–213.
21. A. Rabinovich. Expressive Power of Temporal Logics In Proc. 13th Int. Conf. on Concurrency Theory, vol. 2421 of Lecture Notes in Computer Science, pages 57–75. Springer, 2002.
22. S. Shelah. The monadic theory of order. *Ann. of Math.*, **102**, pp 349-419, 1975.
23. W. Thomas (1997). Ehrenfeucht games, the composition method, and the monadic theory of ordinal words. In *Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht*, *Lecture Notes in Computer Science* **1261**:118-143, Springer-Verlag.
24. T. Wilke. Specifying Time State Sequences in Powerful Decidable Logics and Time Automata. In Formal Techniques in Real Time and Fault Tolerance Systems. LNCS 863 (1994), 694-715.