No Future without (*a hint of*) Past
A Finite Basis for 'Almost Future' Temporal Logic

Dorit Pardo (Ordentlich)*, Alexander Rabinovich*

The Blavatnik School of Computer Science, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.

Abstract

Kamp's theorem established the expressive completeness of the temporal modalities Until and Since for the First-Order Monadic Logic of Order (*FOMLO*) over real and natural time flows. Over natural time, a single future modality (Until) is sufficient to express all future *FOMLO* formulas. These are formulas whose truth value at any moment is determined by what happens from that moment on. Yet this fails to extend to real time domains: here no finite basis of future modalities can express all future *FOMLO* formulas. In this paper we show that finiteness can be recovered if we slightly soften the requirement that future formulas must be totally past-independent: we allow formulas to depend just on the arbitrarily recent past, and maintain the requirement that they be independent of the rest – actually – of most of the past. We call them 'almost future' formulas, and show that there is a finite basis of almost future modalities which is expressively complete (over all Dedekind complete time flows) for the almost future fragment of *FOMLO*.

1. Introduction

Temporal Logic (*TL*) introduced to Computer Science by Pnueli in [Pnu77] is a convenient framework for reasoning about "reactive" systems. This made temporal logics a popular subject in the Computer Science community, enjoying extensive research in the past 30 years. In *TL* we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are expressed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1 \dots \varphi_k$ possibly on 'past' or 'future' points to a statement $\mathsf{M}(\varphi_1 \dots \varphi_k)$ on the 'present' point t_0. The rule to determine the truth of a statement $\mathsf{M}(\varphi_1 \dots \varphi_k)$ at t_0 is called a *Truth Table*. The choice of particular modalities with their truth tables yields different temporal logics. A temporal logic with modalities $\mathsf{M}_1, \dots, \mathsf{M}_k$ is denoted by $TL(\mathsf{M}_1, \dots, \mathsf{M}_k)$.

The simplest example is the one place modality $\mathsf{F}X$ saying: "X holds some time in the future". Its truth table is formalized by $\varphi_{\mathsf{F}}(t_0, X) \equiv (\exists t > t_0)X(t)$. This is a formula of the First-Order Monadic Logic of Order (*FOMLO*) – a fundamental formalism in Mathematical Logic where formulas are built using atomic propositions $P(t)$, atomic relations between elements $t_1 = t_2$, $t_1 < t_2$, Boolean connectives and first-order quantifiers $\exists t$ and $\forall t$. Most modalities used in the literature are defined by such *FOMLO* truth tables, and as a result every temporal formula translates directly into an equivalent *FOMLO* formula. Thus, the different temporal logics may be considered a convenient way to use fragments of *FOMLO*. *FOMLO* can also serve as a yardstick by which to check the strength of temporal logics: A temporal logic is *expressively complete* for a fragment L of *FOMLO* if every formula of L with a single free variable t_0 is equivalent to a temporal formula.

Actually, the notion of expressive completeness is with respect to the type of the underlying model since the question whether two formulas are equivalent depends on the domain over which they are evaluated. Any

*Corresponding author
Email addresses: dorit.pardo@gmail.com (Dorit Pardo (Ordentlich)), rabinoa@post.tau.ac.il (Alexander Rabinovich)

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(partially) ordered set with monadic predicates is a model for *TL* and *FOMLO*, but the main, *canonical*, linear time intended models are the naturals $\langle \mathbb{N}, < \rangle$ for discrete time and the reals $\langle \mathbb{R}, < \rangle$ for continuous time.

A major result concerning *TL* is Kamp's theorem [Kam68, GHR94], which states that the pair of modalities "X *until* Y" and "X *since* Y" is expressively complete for *FOMLO* over the above two linear time canonical models.

Many temporal formalisms studied in computer science concern only future formulas – whose truth value at any moment is determined by what happens from that moment on. For example the formula X *until* Y says that X will hold from now (at least) until a point in the future when Y will hold. The truth value of this formula at a point t_0 does not depend on the question whether $X(t)$ or $Y(t)$ hold at earlier points $t < t_0$.

Over the discrete model $\langle \mathbb{N}, < \rangle$ Kamp's theorem holds also for *future formulas* of *FOMLO*: The future fragment of *FOMLO* has the same expressive power as $TL(\mathsf{Until})$ [GPSS80, GHR94]. The situation is radically different for the continuous time model $\langle \mathbb{R}, < \rangle$. In [HR03] it was shown that $TL(\mathsf{Until})$ is not expressively complete for the future fragment of *FOMLO* and there is no easy way to remedy it. In fact it was shown in [HR03] that there is no temporal logic with a finite set of future modalities which is expressively equivalent to the future fragment of *FOMLO* over the reals.

The proof there goes (roughly) as follows: define a sequence of future formulas $\phi_i(z)$ such that given any set B of modalities definable in the future fragment of *FOMLO* by formulas of quantifier depth at most n, the formula $\phi_{n+1}(z)$ is not expressible in $TL(B)$.

The interesting point is that these formulas are all expressible in a temporal language based on the future modality Until plus the modality K^- of [GHR94]. The formula $\mathsf{K}^-(P)$ holds at a time point t_0 if given any 'earlier' t, no matter how close, we can always come up with a t' in between ($t < t' < t_0$) where P holds. This is of course not a future modality: the formula $\mathsf{K}^-(P)$ is past-dependent. And it turns out that not only the above mentioned sequence of future formulas – but all future formulas – can be expressed (over real time) in $TL(\mathsf{Until}, \mathsf{K}^-)$. This is a consequence of Gabbay's separation theorem [GHR94].

This future-past mixture of Until and K^- is somewhat better than the standard Until-Since basis in the following sense: Although K^- is (like Since) a past modality, it does not depend on much of the past: The formula $\mathsf{K}^-(P)$ depends just on an arbitrarily short 'recent past', and is actually independent of most of the past. In this sense we may say that it is an "almost future" formula (see Section 3.1 for precise definitions).

In [HR03] it was conjectured that $TL(\mathsf{Until}, \mathsf{K}^-)$ is expressively complete for almost future formulas of *FOMLO*. Our main result here affirms this conjecture with respect to all Dedekind complete time domains. An extended abstract concerning the particular case of real time domains was published in [PR12].

The rest of the paper is organized as follows: In Section 2 we recall the definitions of the monadic logic, the temporal logics and Kamp's theorem. In Section 3.1 we define "almost futureness" and make a trivial small step towards the proof. Most of the 'machinery' needed for the proof is in Sections 3.2 and 3.3, with the heart of the proof in Lemma 3.25. Section 3.4 then just puts it all together to complete the proof. Finally, Section 4 states further results and comments.

2. Preliminaries

We start with the basic definitions of First-Order Monadic Logic of Order (***FOMLO***) and Temporal Logic (***TL***), and some well-known results concerning their expressive power. Fix a ***signature*** (finite or infinite) $\mathcal{S}$ of ***atoms***. We use $P, Q, R, S \ldots$ to denote members of $\mathcal{S}$. Syntax and semantics of both logics are defined below with respect to such a fixed signature.

2.1. First-Order Monadic Logic of Order

Syntax: In the context of *FOMLO*, the atoms of $\mathcal{S}$ are referred to (and used) as ***unary predicate symbols***. Formulas are built using these symbols, plus two binary relation symbols, $<$ and $=$, and a finite set of ***first-order variables*** (denoted by $x, y, z, \dots$). Formulas are defined by the grammar:

$$atomic ::= \quad x < y \ \mid \ x = y \ \mid \ P(x) \qquad\quad (\text{where} \ \ P \in \mathcal{S})$$

$$\varphi ::= \quad atomic \ \mid \ \neg\varphi_1 \ \mid \ \varphi_1 \vee \varphi_2 \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \exists x \varphi_1 \ \mid \ \forall x \varphi_1$$

The notation $\varphi(x_1, \dots, x_n)$ implies that φ is a formula where the x_i's are the only variables that may occur free; writing $\varphi(x_1, \dots, x_n, P_1, \dots, P_k)$ additionally implies that the P_i's are the only predicate symbols that may occur in φ. We will also use the standard abbreviated notation for ***bounded quantifiers***, e.g.: $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$, $(\forall x)^{\leq z}(\dots)$ denotes $\forall x((x \leq z) \rightarrow (\dots))$, $(\forall x)^{<u}_{>l}(\dots)$ denotes $\forall x((l < x < u) \rightarrow (\dots))$, etc. Finally, as usual, ***True(x)*** denotes $P(x) \vee \neg P(x)$ and ***False(x)*** denotes $P(x) \wedge \neg P(x)$.

Semantics: Formulas are interpreted over *structures.* A ***structure*** over $\mathcal{S}$ is a triplet $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{T}$ is a set – the ***domain*** of the structure, $<$ is an irreflexive partial order relation on $\mathcal{T}$, and $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{T})$ is the ***interpretation*** of the atoms in the structure (where $\mathcal{P}$ is the powerset notation). We use the standard notation $\mathcal{M}, t_1, t_2, \dots t_n \models \varphi(x_1, x_2, \dots x_n)$. The semantics is defined in the standard way. Notice that for ***formulas with at most one free first-order variable***, this reduces to:

$$\mathcal{M}, t \models \varphi(x).$$

We will often abuse terminology, and shortly refer to such formulas as ***monadic*** formulas (or to the corresponding syntactical fragment – as ***FOMLO***).

2.2. Propositional Temporal Logics

Syntax: In the context of *TL*, the atoms of $\mathcal{S}$ are used as ***atomic propositions*** (also called ***propositional atoms***). Formulas are built using these atoms, and a set (finite or infinite) B of ***modality names***, where a non-negative integer ***arity*** is associated with each $\mathsf{M} \in B$. The syntax of *TL* with the ***basis*** B over the signature $\mathcal{S}$, denoted by ***TL(B)***, is defined by the grammar:

$$F ::= \quad P \ \mid \ \neg F_1 \ \mid \ F_1 \vee F_2 \ \mid \ F_1 \wedge F_2 \ \mid \ \mathsf{M}(F_1, F_2, \dots, F_n)$$

where $P \in \mathcal{S}$ and $\mathsf{M} \in B$ an n-place modality (that is, with arity n). As usual ***True*** denotes $P \vee \neg P$ and ***False*** denotes $P \wedge \neg P$.

Semantics: Formulas are interpreted at ***time-points*** (or ***moments***) in structures (elements of the domain). The domain $\mathcal{T}$ of $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ is called the ***time domain***, and $(\mathcal{T}, <)$ – the ***time flow*** of the structure. The semantics of each n-place modality $\mathsf{M} \in B$ is defined by a 'rule' specifying how the set of moments where $\mathsf{M}(F_1, \dots, F_n)$ holds (in a given structure) is determined by the n sets of moments where each of the formulas F_i holds. Such a 'rule' for M is formally specified by an operator $\mathcal{O}_\mathsf{M}$ on time flows, where given a time flow $\mathcal{F} = (\mathcal{T}, <)$, $\mathcal{O}_\mathsf{M}(\mathcal{F})$ is yet an operator in $(\mathcal{P}(\mathcal{T}))^n \longrightarrow \mathcal{P}(\mathcal{T})$.

The semantics of $TL(B)$ formulas is then defined inductively: Given a structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ and a moment $t \in \mathcal{M}$ (read $t \in \mathcal{M}$ as $t \in \mathcal{T}$), define when a formula F ***holds*** in $\mathcal{M}$ at t, which we denote $\mathcal{M}, t \models F$, as follows:

- $\mathcal{M}, t \models P$ iff $t \in \mathcal{I}(P)$, for any propositional atom P.
- $\mathcal{M}, t \models F \vee G$ iff $\mathcal{M}, t \models F$ or $\mathcal{M}, t \models G$; similarly ("pointwise") for $\wedge$, $\neg$.
- $\mathcal{M}, t \models \mathsf{M}(F_1, \dots, F_n)$ iff $t \in [\mathcal{O}_\mathsf{M}(\mathcal{T}, <)](T_1, \dots, T_n)$ where $\mathsf{M} \in B$ is an n-place modality, $F_1, \dots, F_n$ are formulas and $T_i =_{def} \{s \in \mathcal{T} : \mathcal{M}, s \models F_i\}$.

Truth tables: Most standard modalities studied in the literature can be specified in *FOMLO*: A *FOMLO* formula $\varphi(x, P_1, \ldots, P_n)$ with a single free first-order variable x and with n predicate symbols P_i is called an ***n-place first-order truth table***. Such a truth table φ ***defines*** an n-ary modality M (whose semantics is given by an operator $\mathcal{O}_{\mathsf{M}}$) iff for any time flow $(\mathcal{T}, <)$, for any $T_1, \ldots, T_n \subseteq \mathcal{T}$ and for any structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{I}(P_i) = T_i$:

$$[\mathcal{O}_{\mathsf{M}}(\mathcal{T}, <)](T_1, \ldots, T_n) = \{t \in \mathcal{T} : \mathcal{M}, t \models \varphi(x, P_1, \ldots, P_n)\}$$

Example 2.1. *Below are truth-table definitions for the well-known "**Eventually**" and "**Globally**", the (binary)* ***strict*-Until** *and* ***strict*-Since** *of [Kam68] and for* K^+ *and* K^- *of [GHR94]:*

- $\Diamond$ *("**Eventually**") defined by:* $\varphi_{\Diamond}(x, P) =_{def} (\exists x')_{>x} P(x')$
- $\Box$ *("**Globally**") defined by:* $\varphi_{\Box}(x, P) =_{def} (\forall x')_{>x} P(x')$
- **Until** *defined by :* $\varphi_{\mathsf{Until}}(x, Q, P) =_{def} (\exists x')_{>x}(Q(x') \wedge (\forall y)^{<x'}_{>x} P(y))$
- **Since** *defined by:* $\varphi_{\mathsf{Since}}(x, Q, P) =_{def} (\exists x')^{<x}(Q(x') \wedge (\forall y)^{<x}_{>x'} P(y))$
- K^+ *defined by:* $\varphi_{\mathsf{K}^+}(x, P) =_{def} (\forall x')_{>x}(\exists y)^{<x'}_{>x} P(y)$
- K^- *defined by:* $\varphi_{\mathsf{K}^-}(x, P) =_{def} (\forall x')^{<x}(\exists y)^{<x}_{>x'} P(y)$

The first four modalities above are most commonly denoted by $\mathsf{F}, \mathsf{G}, \mathsf{U}, \mathsf{S}$. We will rather use the old-fashioned notations $\Diamond$ and $\Box$; and we will use infix notation for the binary modalities Until and Since: P Until Q denotes Until(Q, P), meaning "there is some future moment where Q holds, and P holds all along till then". The ***non-strict*** version Until^{ns} is defined as $P \wedge (P \text{ Until } Q)$, requiring that P should hold at the "present moment" as well.

The formula $\mathsf{K}^-(P)$ holds at the "present moment" t_0 iff given any earlier $t < t_0$ – no matter how close – there is a moment t' in between ($t < t' < t_0$) where the formula P holds. Notice that K^+ and K^- are definable in terms of Until and Since:

$$\mathsf{K}^+(P) \equiv \neg(\neg P \text{ Until } True)$$

$$\mathsf{K}^-(P) \equiv \neg(\neg P \text{ Since } True)$$

In [GHR94] these two modalities are actually defined as abbreviations in *TL*(Until, Since).

2.3. Kamp's Theorem

We are interested in the relative expressive power of *TL* (compared to *FOMLO*) over the class of ***linear structures***. Major results in this area are with respect to the subclass of ***Dedekind complete structures*** – where the order is Dedekind complete, that is, where every non empty subset (of the domain) which has an upper bound has a least upper bound.

Equivalence between temporal and monadic formulas is naturally defined: $F \equiv \varphi(x)$ iff for any $\mathcal{M}$ and $t \in \mathcal{M}$: $\mathcal{M}, t \models F \Leftrightarrow \mathcal{M}, t \models \varphi(x)$. We will occasionally write $\equiv_{\mathcal{L}}$, $\equiv_{\mathcal{DC}}$, $\equiv_{\mathcal{C}}$ to distinguish equivalence over linear structures, over Dedekind complete structures, over any class $\mathcal{C}$ of structures, respectively.

Definability: A temporal modality is definable in *FOMLO* iff it has a *FOMLO* truth table; a temporal formula F is definable in *FOMLO* over a class $\mathcal{C}$ of structures iff there is a monadic formula $\varphi(z)$ such that $F \equiv_{\mathcal{C}} \varphi(z)$. In this case we say that φ ***defines*** F over $\mathcal{C}$. Similarly, a monadic formula $\varphi(z)$ may be definable in $TL(B)$ over $\mathcal{C}$.

Expressive completeness/ equivalence: A temporal language $TL(B)$ (as well as the basis B) is expressively complete for (a fragment of) *FOMLO* over a class $\mathcal{C}$ of structures iff all monadic formulas (of that fragment) $\varphi(z)$ are definable over $\mathcal{C}$ in $TL(B)$. Similarly, one may speak of expressive completeness of *FOMLO* for some temporal language. If we have expressive completeness in both directions between two languages – they are ***expressively equivalent***.

As Until and Since are definable in *FOMLO*, it follows that *FOMLO* is expressively complete for *TL*(Until, Since). The fundamental theorem of Kamp shows that for Dedekind complete structures the opposite direction holds as well:

Theorem 2.2 ([Kam68]). *TL*(Until, Since) *is expressively equivalent to FOMLO over Dedekind complete structures.*

This was further generalized by Stavi who introduced two new modalities Until′ and Since′ and proved that *TL*(Until, Since, Until′, Since′) and *FOMLO* have the same expressive power over all linear time flows [GPSS80, GHR94].

2.4. In Search of a Finite Basis for Future Formulas

We use standard ***interval*** notations and terminology for subsets of the domain of a structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$, e.g.: $(t, \infty) =_{def} \{t' \in \mathcal{T} | t' > t\}$; similarly we define $(t, t'), [t, t'), (t, \infty), [t, \infty)$, etc., where $t < t'$ are the ***endpoints*** of the interval. The ***sub-structure*** of $\mathcal{M}$ restricted to an interval is defined naturally. In particular: $\mathcal{M}|_{>t_0}$ denotes the sub-structure of $\mathcal{M}$ restricted to (t_0, ∞): Its domain is (t_0, ∞) and its order relation and interpretation are those of $\mathcal{M}$, restricted to this interval. $\mathcal{M}|_{\geq t_0}$ is defined similarly with respect to $[t_0, \infty)$. Notice that if $\mathcal{M}$ is Dedekind complete then so is any sub-structure obtained by restricting $\mathcal{M}$ to an interval of its domain. If structures $\mathcal{M}, \mathcal{M}'$ have domains $\mathcal{T}, \mathcal{T}'$, and if I is an interval of $\mathcal{M}$, with endpoints $t_1 < t_2$ in $\mathcal{M}$, such that $I \cup \{t_1, t_2\} \subseteq \mathcal{T} \cap \mathcal{T}'$ and the order relations of both structures coincide on $I \cup \{t_1, t_2\}$ – we will say that I is a ***common interval*** of both structures. This is defined similarly for intervals with ∞ or $-\infty$ as either endpoint. Two structures ***coincide*** on a common interval iff the interpretations coincide there. Two structures ***agree*** on a formula at a given common time-point (or along a common interval) iff the formula has the same truth value at that point (or along that interval) in both structures.

Definition 2.3 (Future / past formulas and modalities). *A formula (temporal, or monadic with at most one free variable) F is* ***(semantically)****:*

- *A* ***future*** *formula iff whenever two linear structures coincide on a common interval $[t_0, \infty)$ they agree on F at t_0.*
- *A* ***pure future*** *formula iff whenever two linear structures coincide on a common interval (t_0, ∞) they agree on F at t_0.*

Past *and* ***pure past*** *formulas are defined similarly. A temporal modality is a first-order* ***future (past) modality*** *iff it is definable in FOMLO by a future (past) truth table.*

Note that 'future' can be characterized also as follows: A formula $\varphi(x_0)$ is a future formula iff it is equivalent to a formula with all quantifiers relativized to $[x_0, \infty)$, that is, all quantifiers are of the form $(\forall x)_{\geq x_0}(\dots)$ or $(\exists x)_{\geq x_0}(\dots)$.

Looking at their truth tables, it is easy to verify that Until is a future modality and Since is a past modality. This pair {Until, Since} forms an expressively complete (finite) basis in the sense of Kamp's theorem. Do we have a finite basis of future modalities which is expressively complete for all future formulas? Here are some answers:

Theorem 2.4 ([GPSS80]). *TL*(Until) *is expressively equivalent to the future fragment of FOMLO over discrete time flows (naturals, integers, finite).*

Theorem 2.5 ([HR03]). *There is no temporal logic with a finite basis of* ***future*** *modalities which is expressively equivalent to the future fragment of FOMLO over real time flows.*

Theorem 2.6 ([GHR94], [Ra14]). *TL*(Until, K^-) *is expressively complete for the future fragment of FOMLO over Dedekind complete time flows.*

In Theorem 2.6 we don't have expressive equivalence, as not all $TL(\mathsf{Until}, \mathsf{K}^-)$ formulas are future formulas. This theorem offers a finite basis $\{\mathsf{Until}, \mathsf{K}^-\}$, but just like Kamp's $\{\mathsf{Until}, \mathsf{Since}\}$ – this is a 'mixed' future-past basis. [HR03] points out that in spite of its 'past' nature, K^- is *almost* a future modality because it depends just on an arbitrarily small portion of the recent past, and is independent of most of the past. It is conjectured there that this "almost future" basis generates only such "almost future" formulas, and that it generates ***all*** of them. In this paper we show that this conjecture holds over all Dedekind complete time domains.

Although not explicitly stated in [GHR94], Theorem 2.6 easily follows their work along the proof of their separation theorem[1]. Another proof for Theorem 2.6 can be found in [Ra14][2].

3. A Finite Basis for Almost Future Formulas over Dedekind Complete Time

In Section 3.1 below we give a formal definition for "almost futureness" and formulate the main result (Theorem 3.11). Section 3.2 "decomposes" monadic formulas to reach the very specific normal form of Corollary 3.21. Then the most technical part of the proof is in Section 3.3, with the heart of the proof in Lemma 3.25 which, eventually, will allow the translation of this normal form into $TL(\mathsf{Until}, \mathsf{K}^-)$. Section 3.4 finally puts it all together to complete the proof.

3.1. Almost Future Formulas

Definition 3.1 (Almost future formulas, modalities, bases). *A formula (temporal, or monadic with at most one free variable) F is an almost future formula iff whenever two linear structures coincide on a common interval (t, ∞) they agree on F all along (t, ∞). A temporal modality is almost future iff it has an almost future truth table in FOMLO. A basis is almost future iff all its modalities are.*

Clearly, all pure future formulas are in particular future formulas and all future formulas are almost future. Note that we can give an alternative (equivalent) definition for future and pure future formulas in the style of Definition 3.1 as follows (compare with Definition 2.3): A formula F is

- ***Future*** iff whenever two linear structures coincide on a common interval $[t, \infty)$ they agree on F all along $[t, \infty)$.
- ***Pure future*** iff whenever two linear structures coincide on a common interval (t, ∞) they agree on F all along $[t, \infty)$.

Remark 3.2. *The next two facts and the lemma below follow immediately:*

1. *If an almost future formula holds at t_0 in a substructure $\mathcal{M}|_{\geq t}$ of a linear structure $\mathcal{M}$ where $t < t_0$ then it holds there in $\mathcal{M}$ as well.*
2. *If an almost future formula holds at t_0 in a linear structure $\mathcal{M}$ then it holds at t_0 in all substructures $\mathcal{M}|_{\geq t}$, where $t < t_0$.*

Lemma 3.3. *If a basis B is almost future then so are all of $TL(B)$ formulas. In particular:* Until, K^- *and all the formulas of $TL(\mathsf{Until}, \mathsf{K}^-)$ are almost future.*

Example 3.4. *Consider a structure with a real time domain and a point t_0 that satisfies the condition: "Any open interval (t, t_0) contains a proper subinterval (t_2, t_1) such that P (an atomic property) holds at the ends t_1 and t_2, but doesn't hold anywhere inside (t_2, t_1)". This is an almost future property expressible in FOMLO. In $TL(\mathsf{Until}, \mathsf{K}^-)$ it is expressed by the following formula (evaluated at t_0):*

$$\mathsf{K}^-(P \wedge (\neg P \ \mathsf{Until}\ P))$$

[1] [GHR94] Theorem 10.3.20.
[2] [Ra14] Theorem 7.4

It turns out that studying "almost futureness" requires a close examination of how the recent past looks like. Considering various kinds of linear time flows, we classify time points according to their recent past:

Definition 3.5. *Given a linear structure $\mathcal{M}$: If $\mathcal{M}$ has a minimal element we call it a **first moment** in $\mathcal{M}$. If $t_1 < t_2$ in $\mathcal{M}$ and the interval (t_1, t_2) is empty we say that t_2 is the **successor** of t_1 in $\mathcal{M}$ and t_1 is the **predecessor** of t_2 in $\mathcal{M}$. If $t \in \mathcal{M}$ is not a first moment in $\mathcal{M}$ and for any $t' < t$ the interval (t', t) is non-empty we say that t is a **left-limit** in $\mathcal{M}$.*

The natural time flow $\langle \mathbb{N}, < \rangle$, for example, demonstrates successors, predecessors and a first moment; while in the real time flow $\langle \mathbb{R}, < \rangle$ all moments are left-limits. The following example illustrates this classification, as well as the subtleties of 'almost future' and the expressivity of $TL(\mathsf{Until}, \mathsf{K}^-)$.

Example 3.6. *The formulas **FM, SC, LL** defined below are almost future (and let **fm(z), sc(z), ll(z)** denote particular, equivalent FOMLO versions, respectively); But **Pred(Q)** below (with Q atomic) is not almost future (compare it with the almost future SC):*

$FM =_{def} \mathsf{K}^-(False)$	*"I am a first moment"*
$SC =_{def} \neg\mathsf{K}^-(True)$	*"I am a successor"*
$LL =_{def} \neg\mathsf{K}^-(False) \wedge \mathsf{K}^-(True)$	*"I am a left-limit"*
$Pred(Q) =_{def} (False \ \mathsf{Since} \ Q)$	*"My predecessor is Q"*
$Pred(True) = (False \ \mathsf{Since} \ True)$	*Equivalent to the almost future SC !*

Remark 3.7. *Any point in any linear structure is (exclusively) either a first moment or a successor or a left-limit. These three 'types' of points are indeed characterized by the corresponding $TL(\mathsf{Until}, \mathsf{K}^-)$ formulas:*
$\mathcal{M}, t \models FM$ iff t is a first moment in $\mathcal{M}$.
$\mathcal{M}, t \models SC$ iff t is a successor in $\mathcal{M}$.
$\mathcal{M}, t \models LL$ iff t is a left-limit in $\mathcal{M}$.

Remark 3.8. *As first moments have actually no past, formulas 'behave' at first moments as if they were future formulas: for every FOMLO formula $\psi(z)$ there is a future formula $\overset{\mapsto}{\psi}(z)$ such that*

$$\psi(z) \wedge fm(z) \equiv_{\mathcal{L}} \overset{\mapsto}{\psi}(z) \wedge fm(z)$$

Take for $\overset{\mapsto}{\psi}$ the 'relativised' version of ψ, where all quantifiers are bounded by '$\geq z$'.

Almost future formulas demonstrate a similar phenomenon with respect to successor moments: successors do have a past; but we may say that they have no recent past: the entire past of a successor resides 'beyond' its predecessor. Thus, the evaluation of an almost future formula to $True$ or $False$ at a successor point depends only on the present and future. Formally:

Lemma 3.9. *For every almost future FOMLO formula $\psi(z)$ there is a future formula $\overset{\rightarrow}{\psi}(z)$ such that*

$$\psi(z) \wedge sc(z) \equiv_{\mathcal{L}} \overset{\rightarrow}{\psi}(z) \wedge sc(z)$$

To prove this lemma we 'borrow' the following from the work of [GHR94] concerning separation of formulas into past, present and future formulas:

Proposition 3.10 ([GHR94] Semantical Separation in *FOMLO*). *Every FOMLO formula* $\psi(z)$ *is equivalent over linear time to a finite disjunction of the form:* $\bigvee_i(\pi_i(z) \wedge \gamma_i(z) \wedge \varphi_i(z))$ *with pure past, present, pure future FOMLO formulas* π_i, γ_i, φ_i, *respectively.*

The proposition and a hint to its proof is embedded in the proof of [GHR94] Theorem 9.3.4.

Proof. *(of Lemma 3.9)* Given almost future $\psi(z)$, by Proposition 3.10 we have $\psi(z) \equiv_{\mathcal{L}} \bigvee(\pi_i(z) \wedge \varphi_i(z))$ with pure past π_i and future (possibly non-pure) φ_i. Denote: $good =_{def} \{i : \pi_i(z)$ is *consistent* with $sc(z)\}$; by *consistent* here we mean that the conjunction $\pi_i(z) \wedge sc(z)$ is satisfiable. Now denote:

$$\vec{\psi}(z) =_{def} \bigvee_{i \in good} \varphi_i(z)$$

Then $\psi(z) \wedge sc(z) \equiv_{\mathcal{L}} \vec{\psi}(z) \wedge sc(z)$. The $\Rightarrow$ direction is trivial. For the $\Leftarrow$ direction assume that $\mathcal{M}, t \models \varphi_i(z) \wedge sc(z)$ for some $i \in good$ and a linear structure $\mathcal{M}$. Then the corresponding π_i is *consistent* with sc, i.e., $\mathcal{M}', t' \models \pi_i(z) \wedge sc(z)$ for some $\mathcal{M}', t'$ (where t' is a successor moment). Replace the interval $[t', \infty)$ of $\mathcal{M}'$ by the corresponding interval $[t, \infty)$ of $\mathcal{M}$. Denote the structure obtained this way by $\mathcal{M}''$. Then $\mathcal{M}'', t \models \pi_i(z)$ because π_i is a pure past formula. Also, as φ_i is a future formula, independent of past – $\mathcal{M}'', t \models \varphi_i(z)$. So we have $\mathcal{M}'', t \models \pi_i(z) \wedge \varphi_i(z)$, hence $\mathcal{M}'', t \models \psi(z)$. Finally, as $\psi(z)$ is almost future, and as $\mathcal{M}$ and $\mathcal{M}''$ coincide on the common interval (t_1, ∞) where t_1 is the predecessor[3] of t – $\mathcal{M}, t \models \psi(z)$ as well. □

We have seen some examples of almost future properties expressed in $TL(\mathsf{Until}, \mathsf{K}^-)$. Our main result states that ***any*** almost future property expressible in *FOMLO* can be translated to $TL(\mathsf{Until}, \mathsf{K}^-)$:

Main Theorem 3.11. $TL(\mathsf{Until}, \mathsf{K}^-)$ *is expressively equivalent to the almost future fragment of FOMLO over Dedekind complete structures.*

As Until and K^- are definable in *FOMLO*, the expressive completeness of almost future *FOMLO* for $TL(\mathsf{Until}, \mathsf{K}^-)$ over all linear structures (and in particular over Dedekind complete ones) follows immediately by Lemma 3.3. For the opposite direction we have to show how almost future monadic formulas translate into $TL(\mathsf{Until}, \mathsf{K}^-)$.

Remark 3.8 and Lemma 3.9 suggest two semantics-preserving translations: over first moments (for all formulas) and over successor moments (for almost future formulas). Indeed, over Dedekind complete structures:

Lemma 3.12 (Translations over first moments and successors). *There are translations* $Trans^{fm}$ *and* $Trans^{sc}$ *from FOMLO into* $TL(\mathsf{Until}, \mathsf{K}^-)$*, such that for any* $\varphi(z)$ *and for any almost future* $\psi(z)$*:*

$$\varphi(z) \wedge fm(z) \equiv_{\mathcal{DC}} Trans^{fm}(\varphi) \wedge FM$$
$$\psi(z) \wedge sc(z) \equiv_{\mathcal{DC}} Trans^{sc}(\psi) \wedge SC$$

where the formulas FM, SC, fm, sc *are as defined in Example 3.6*

Proof. Immediate by Remark 3.8, Lemma 3.9 and Theorem 2.6. □

Most of our effort will now be in finding a translation for left-limit points.

[3] Identifying the predecessor of t in $\mathcal{M}$ with its predecessor in $\mathcal{M}''$ – a harmless abuse of notations.

3.2. Decomposition

Expressive completeness proofs often go through manipulating formulas to reach some (equivalent) standard form that can then be translated to the target language. Here we start with a special case of 'decomposition formulas' used by [GPSS80] in their proof for Theorem 2.4. This will be further 'decomposed' to reach the normal form of Corollary 3.21, which plays a key role in the proof of our main result.

Definition 3.13 ([GPSS80] ∃∀-formulas).

1. *An* ***∃∀-formula*** *is a formula of the following form, where $n, m \geq 0$ and $\alpha_i, \beta_i, \alpha_i^*, \beta_i^*$ are FOMLO formulas with (at most) a single free first-order variable:*

$$\begin{aligned}
\psi(z) := {} & \exists p_0 \exists p_1 \dots \exists p_n \exists f_0 \exists f_1 \dots \exists f_m \\
& [(p_n < p_{n-1} < \cdots < p_1 < p_0 = z = f_0 < f_1 < f_2 \cdots < f_m) && \text{“Ordering”} \\
& \wedge \bigwedge_{i=0}^{n} \alpha_i(p_i) && \text{“All } \alpha_i\text{'s held at the points } p_i \ (\alpha_0 \text{ holds at } z)\text{”} \\
& \wedge (\forall y)^{<p_n} \beta_n(y) && \text{“}\beta_n \text{ held ‘ever before’ } p_n\text{”} \\
& \wedge \bigwedge_{i=0}^{n-1} [(\forall y)^{<p_i}_{>p_{i+1}} \beta_i(y)] && \text{“Each } \beta_i \text{ held along } (p_{i+1}, p_i)\text{”} \\
& \wedge \bigwedge_{i=1}^{m} \alpha_i^*(f_i) && \text{“All } \alpha_i^*\text{'s will hold at the points } f_i\text{”} \\
& \wedge \bigwedge_{i=0}^{m-1} [(\forall y)^{<f_{i+1}}_{>f_i} \beta_i^*(y)] && \text{“Each } \beta_i^* \text{ will hold along } (f_i, f_{i+1})\text{”} \\
& \wedge (\forall y)_{>f_m} \beta_m^*(y)] && \text{“}\beta_m^* \text{ will hold held ‘ever after’ } f_m\text{”}
\end{aligned}$$

2. *An* ***$\overleftarrow{\exists}\forall$-formula*** *is an ∃∀-formula as above, where $m = 0$ and $\beta_0^* = True$.*
3. *Given a temporal basis B, an* ***∃∀-formula over TL(B)*** *is an ∃∀-formula where the formulas involved ($\alpha_i, \beta_i, \alpha_i^*, \beta_i^*$) are all definable over Dedekind complete structures in $TL(B)$.*
4. *Similarly, given a basis B, an* ***$\overleftarrow{\exists}\forall$-formula over TL(B)*** *is an $\overleftarrow{\exists}\forall$-formula where α_i, β_i are all definable over Dedekind complete structures in $TL(B)$.*

Notice that the first two definitions above are purely syntactical. p_0 and f_0 are dummy variables, introduced here to simplify notations.

Notation 3.14. *We use the following abbreviated notation for ∃∀-formulas and for $\overleftarrow{\exists}\forall$-formulas $\psi(z)$ as above:*

$$\psi(z) = (\langle\beta_n, \alpha_n\rangle, \dots, \langle\beta_1, \alpha_1\rangle, \langle\beta_0, \alpha_0, \beta_0^*\rangle, \langle\alpha_1^*, \beta_1^*\rangle, \dots, \langle\alpha_m^*, \beta_m^*\rangle) \qquad (\exists\forall)$$

$$\psi(z) = (\langle\beta_n, \alpha_n\rangle, \dots, \langle\beta_1, \alpha_1\rangle, \langle\beta_0, \alpha_0\rangle) \qquad (\overleftarrow{\exists}\forall)$$

Notice that we omit β_0^* (True) in $\overleftarrow{\exists}\forall$-formulas. We will occasionally write

$$\psi^n(z) = (\langle\beta_n, \alpha_n\rangle, \dots, \langle\beta_0, \alpha_0\rangle)$$

to explicitly reflect the length of the quantifier prefix of an $\overleftarrow{\exists}\forall$-formula.

The next lemma is an instance of Lemma 3.2 (1) of [Ra14]: the more general context there involves a form similar to ∃∀-formulas, but with any number of free first-order variables $z_0, z_1, \ldots, z_k$.

Lemma 3.15. *Given a temporal basis B, any finite conjunction of ∃∀-formulas over $TL(B)$ is equivalent to a finite disjunction of ∃∀-formulas over $TL(B)$.*

PROOF. (Hint). Consider, for example, ψ and ψ' below, with atomic $p_i, q_i, r_i, p'_i, q'_i, r'_i$:

$$\psi(z) = (\langle q_1, p_1\rangle, \langle q_0, p_0, r_0\rangle)$$
$$\psi'(z) = (\langle q'_1, p'_1\rangle, \langle q'_0, p'_0, r'_0\rangle)$$

It is easy to verify that $\psi(z) \wedge \psi'(z) \equiv_{\mathcal{L}} \delta_1(z) \vee \delta_2(z) \vee \delta_3(z)$, where

$$\delta_1(z) = (\langle q_1 \wedge q'_1, p_1 \wedge q'_1\rangle, \langle q_0 \wedge q'_1, q_0 \wedge p'_1\rangle, \langle q_0 \wedge q'_0, p_0 \wedge p'_0, r_0 \wedge r'_0\rangle)$$
$$\delta_2(z) = (\langle q_1 \wedge q'_1, p_1 \wedge p'_1\rangle, \langle q_0 \wedge q'_0, p_0 \wedge p'_0, r_0 \wedge r'_0\rangle)$$
$$\delta_3(z) = (\langle q_1 \wedge q'_1, q_1 \wedge p'_1\rangle, \langle q_1 \wedge q'_0, p_1 \wedge q'_0\rangle, \langle q_0 \wedge q'_0, p_0 \wedge p'_0, r_0 \wedge r'_0\rangle)$$

Extending this example to the general case is straight forward. Notice also that, given a basis B, if ψ and ψ' are both ∃∀-formulas over $TL(B)$ then so are the disjuncts constructed as illustrated above. □

Unsurprisingly, with the presence of Until, the above two forms – ∃∀-formulas and $\overleftarrow{\exists\forall}$-formulas – are equally expressive:

Lemma 3.16. *Let B be any temporal basis such that* Until $\in B$. *Then every ∃∀-formula over $TL(B)$ is equivalent (over all linear structures) to an $\overleftarrow{\exists\forall}$-formula over $TL(B)$.*

PROOF. Let $\delta(z) = (\langle\beta_n, \alpha_n\rangle, \ldots, \langle\beta_1, \alpha_1\rangle, \langle\beta_0, \alpha_0, \beta^*_0\rangle, \langle\alpha^*_1, \beta^*_1\rangle, \ldots, \langle\alpha^*_m, \beta^*_m\rangle)$ be an ∃∀-formula over $TL(B)$, and let A_i, B_i, A^*_i, B^*_i be $TL(B)$ formulas defining $\alpha_i, \beta_i, \alpha^*_i, \beta^*_i$, respectively. Let ψ be the following $\overleftarrow{\exists\forall}$-formula:

$$\psi(z) =_{def} (\langle\beta_n, \alpha_n\rangle, \ldots, \langle\beta_1, \alpha_1\rangle, \langle\beta_0, \alpha^*_0\rangle)$$

where:

$$\begin{aligned}\alpha^*_0(x) =_{def}\ & \alpha_0(x) \wedge \exists f_0 \exists f_1 \ldots \exists f_m[(x = f_0 < f_1 < f_2 \cdots < f_m) \\ & \wedge \bigwedge_{i=1}^{m} \alpha^*_i(f_i) \\ & \wedge \bigwedge_{i=0}^{m-1} [(\forall y)^{<f_{i+1}}_{>f_i} \beta^*_i(y)] \\ & \wedge (\forall y)_{>f_m} \beta^*_m(y)]\end{aligned}$$

Then clearly, $\delta(z) \equiv_{\mathcal{L}} \psi(z)$, and:

$$\alpha^*_0(x) \equiv_{\mathcal{DC}} A_0 \wedge \big[B^*_0 \text{ Until } (A^*_1 \wedge (B^*_1 \text{ Until } (A^*_2 \wedge (B^*_2 \text{ Until } \ldots \text{ Until } (A^*_{m-1} \wedge (B^*_{m-1} \text{ Until } [A^*_m \wedge \neg(True \text{ Until } \neg B^*_m)])) \ldots)))))\big]$$

Hence α^*_0 is definable over Dedekind complete structures in $TL(B)$ and thus ψ above is an $\overleftarrow{\exists\forall}$-formula over $TL(B)$. □

While ∃∀-formulas provide a convenient decomposition over Kamp's basis {Until, Since}, this no longer holds if we restrict the temporal basis to {Until, K^-}: in this case, as mentioned in [Ra14], a specific form of 'unbounded sentences' remains inexpressible by ∃∀-formulas. These are sentences that assert that "π is unbounded from below" for some property π:

Definition 3.17 ([Ra14] "Unbounded from Below" Sentences).

1. *An **unbounded sentence** is a sentence of the form:*

$$\eta := \forall x (\exists y)^{<x} \pi(y)$$

where π is any FOMLO formula with (at most) a single free variable.

2. *Given any temporal basis B, an **unbounded sentence over TL(B)** is an unbounded sentence as above, where the formula π is definable over Dedekind complete structures in $TL(B)$.*

[Ra14] shows that every *FOMLO* formula is equivalent over Dedekind complete structures to a positive boolean combination of unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$ and a generalized form of $\exists\forall$-formulas (with possibly more than one free variable) over $TL(\mathsf{Until}, \mathsf{K}^-)$. The next proposition is an instance of this result – a restricted version concerning formulas with a single free variable.

Proposition 3.18 ([Ra14]). *Every FOMLO formula with (at most) a single free first-order variable is equivalent over Dedekind complete structures to a positive boolean combination of $\exists\forall$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$ and unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$.*

Consider a formula $\varphi(z)$ represented by the above proposition as:

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_i \bigwedge_j \varphi_{ij}(z) \tag{1}$$

where each φ_{ij} is either an unbounded sentence over $TL(\mathsf{Until}, \mathsf{K}^-)$ or an $\exists\forall$-formula over $TL(\mathsf{Until}, \mathsf{K}^-)$. Some of the disjuncts here may be 'pure' – consisting of $\exists\forall$-conjuncts only, while others may be 'mixed' – including at least one 'unbounded' conjunct. And we can assume without loss of generality that at least one disjunct is 'pure': otherwise, add to (1) a dummy disjunct in the form of an unsatisfiable $\exists\forall$-formula, for example $(\langle True, False, True\rangle)$. Thus, (1) above with $k \geq 1$ 'pure' disjuncts can be rearranged as:

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_{i=1}^{k} \bigwedge_j \psi_{ij}(z) \vee \bigvee_i \Big(\eta_i \wedge \bigwedge_j \chi_{ij}(z)\Big)$$

where ψ_{ij} are all $\exists\forall$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$, η_i are all unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$ and χ_{ij} are of either type. And by Lemma 3.15 and Lemma 3.16, conclude:

Corollary 3.19. *Every FOMLO formula $\varphi(z)$ with (at most) a single free first-order variable is equivalent over Dedekind complete structures to a formula in the following form:*

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_{i=1}^{k} \psi_i(z) \vee \bigvee_i \Big(\eta_i \wedge \bigwedge_j \chi_{ij}(z)\Big)$$

where $k \geq 1$, ψ_i are all $\overleftarrow{\exists}\forall$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$, η_i are all unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$ and χ_{ij} are of either type.

Now, consider an $\overleftarrow{\exists}\forall$-formula $\psi^n(z) = (\langle \beta_n, \alpha_n\rangle, \dots, \langle \beta_0, \alpha_0\rangle)$. Clearly, for any arbitrary formulas $\chi_1(x)$ and $\chi_2(x)$:

$$\psi^n(z) \equiv_{\mathcal{L}} (\langle \chi_1, \alpha_n \wedge fm\rangle, \dots, \langle \beta_0, \alpha_0\rangle) \vee (\langle \beta_n, \alpha_n \wedge \neg fm\rangle, \dots, \langle \beta_0, \alpha_0\rangle) \equiv_{\mathcal{L}}$$
$$(\langle \chi_1, \alpha_n \wedge fm\rangle, \dots, \langle \beta_0, \alpha_0\rangle) \;\vee$$
$$(\langle \beta_n \wedge \neg fm, \alpha_n \wedge \neg fm\rangle, \dots, \langle \beta_0, \alpha_0\rangle) \;\vee$$
$$(\langle \chi_2, \beta_n \wedge fm\rangle, \langle \beta_n \wedge \neg fm, \alpha_n \wedge \neg fm\rangle, \dots, \langle \beta_0, \alpha_0\rangle)$$

where ***fm(x)*** is the monadic formula characterizing first moments (recall Example 3.6). In particular, let $\chi_1 = \chi_2 = True$ and then by Remark 3.8:

$$\begin{aligned}\psi^n(z) \equiv_{\mathcal{L}} & \left(\langle True, \alpha'_n \wedge fm\rangle, \ldots, \langle \beta_0, \alpha_0\rangle\right) \vee \\ & \left(\langle \beta_n \wedge \neg fm, \alpha_n \wedge \neg fm\rangle, \ldots, \langle \beta_0, \alpha_0\rangle\right) \vee \\ & \left(\langle True, \beta'_n \wedge fm\rangle, \langle \beta_n \wedge \neg fm, \alpha_n \wedge \neg fm\rangle, \ldots, \langle \beta_0, \alpha_0\rangle\right)\end{aligned} \tag{2}$$

where α'_n, β'_n are future formulas. And notice that if $\psi^n(z)$ is an $\overleftarrow{\exists\forall}$-formula over $TL(\mathsf{Until}, \mathsf{K}^-)$, then (2) above is a disjunction of three $\overleftarrow{\exists\forall}$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$: α'_n and β'_n are definable over Dedekind complete structures in $TL(\mathsf{Until}, \mathsf{K}^-)$ by Theorem 2.6; and the formulas fm and $True$ are clearly definable in $TL(\mathsf{Until}, \mathsf{K}^-)$ as well. Notice also that the first and third disjuncts in (2) have a leftmost component in the form: $\langle True, \textit{future-formula} \wedge fm\rangle$, while the second disjunct has a leftmost component in the form: $\langle \beta \wedge \neg fm, \alpha \wedge \neg fm\rangle$.

Corollary 3.19 can now be refined, using these observations, to reach the normal form of Corollary 3.21 below. First, we introduce the necessary definitions:

Definition 3.20 (FM-$\overleftarrow{\exists\forall}$-formulas and non-FM-$\overleftarrow{\exists\forall}$-formulas).

1. *An **FM-$\overleftarrow{\exists\forall}$-formula** is an $\overleftarrow{\exists\forall}$-formula of the form:*

$$\psi^n = \left(\langle True, \alpha_n \wedge fm\rangle, \langle \beta_{n-1}, \alpha_{n-1}\rangle, \ldots, \langle \beta_0, \alpha_0\rangle\right)$$

   where α_i, β_i are all definable over Dedekind complete structures in $TL(\mathsf{Until}, \mathsf{K}^-)$, α_n is a future formula, and fm is the monadic formula characterizing first moments (see Example 3.6).

2. *A **non-FM-$\overleftarrow{\exists\forall}$-formula** is an $\overleftarrow{\exists\forall}$-formula of the form:*

$$\psi^n = \left(\langle \beta_n \wedge \neg fm, \alpha_n \wedge \neg fm\rangle, \langle \beta_{n-1}, \alpha_{n-1}\rangle, \ldots, \langle \beta_0, \alpha_0\rangle\right)$$

   where α_i, β_i are all definable over Dedekind complete structures in $TL(\mathsf{Until}, \mathsf{K}^-)$, and fm as above.

We will refer to these two specific types of $\overleftarrow{\exists\forall}$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$ as ***FM-type*** and ***non-FM-type***. Notice that, in general, $\overleftarrow{\exists\forall}$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$ don't have to be of either type. For example: $\psi^1 = (\langle \beta, \alpha\rangle, \langle \beta, \alpha\rangle)$ with $\alpha(x) = (\forall x')^{<x}(\exists y)^{<x}_{>x'} P(y)$ and $\beta(x) = Q(x)$ (where P, Q are atoms) is of neither type. But, as we have just seen, every $\overleftarrow{\exists\forall}$-formula over $TL(\mathsf{Until}, \mathsf{K}^-)$ is equivalent to a disjunction of FM and non-FM $\overleftarrow{\exists\forall}$-formulas. Hence, from Corollary 3.19 we derive:

Corollary 3.21. *Every FOMLO formula $\varphi(z)$ with (at most) a single free first-order variable is equivalent over Dedekind complete structures to a formula in the following form:*

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_{i=1}^{k} \psi_i(z) \vee \bigvee_i \Big(\eta_i \wedge \bigwedge_j \chi_{ij}(z)\Big) \tag{3}$$

where $k \geq 1$, ψ_i are all $\overleftarrow{\exists\forall}$-formulas of type FM or non-FM, η_i are all unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$ and χ_{ij} are either unbounded sentences over $TL(\mathsf{Until}, \mathsf{K}^-)$ or $\overleftarrow{\exists\forall}$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$.

In the following sections we will see that what really matters in (3) above is the disjunction $\bigvee \psi_i(z)$ of FM-type and non-FM type $\overleftarrow{\exists\forall}$-formulas.

3.3. Formulas that Hold "Regardless of Most of the Past"

A formula F "holds in $\mathcal{M}$ at t_0 regardless of most of the past" if we can truncate the past as close as we wish to the left of t_0, and F persistently holds at t_0 in all such truncated structures. To simplify terminology we will just say that, in such case, F "almost-holds in $\mathcal{M}$ at t_0". Formally:

Definition 3.22 ('Almost-holding'). *Given a formula (temporal, or monadic with at most one free variable) F, and given a linear structure $\mathcal{M}$ and a left-limit moment $t_0 \in \mathcal{M}$, we will say that F* ***almost-holds*** *in $\mathcal{M}$ at t_0 iff for every $t < t_0$ in $\mathcal{M}$ there is a $t' \in (t, t_0)$ such that $\mathcal{M}|_{\geq t'}, t_0 \models F$.*

Remark 3.23.

1. *If F is an almost future formula and t_0 is a left-limit moment in a linear structure $\mathcal{M}$ then: F holds in $\mathcal{M}$ at t_0 iff it almost-holds there.*
2. *In general, it might be the case that a formula F (which is not almost future) almost-holds in $\mathcal{M}$ at a left-limit moment t_0, yet F does not hold in $\mathcal{M}$ at t_0. Examples: "P always held in the past" ($(\forall x)^{<z}P(x)$), "There was a first moment" ($\exists x((x < z) \wedge \forall y(x < y))$). Similarly, "P once held in the past" ($(\exists x)^{<z}P(x)$) demonstrates the converse situation.*
3. *Notice also that, by definition, non-FM $\overleftarrow{\exists\forall}$-formulas and unbounded sentences are unsatisfiable in structures that have a first moment, and therefore don't almost-hold anywhere. Similarly, 'short' FM $\overleftarrow{\exists\forall}$-formulas of the form $\psi^0 = (\langle True, \alpha_0 \wedge fm\rangle)$ are unsatisfiable over left-limits, and thus cannot almost-hold anywhere.*

Lemma 3.24. *If a finite disjunction of FOMLO formulas $\varphi(z) = \bigvee \delta_i(z)$ is almost future, then for every left-limit moment t_0 in every linear structure $\mathcal{M}$:*

$$\mathcal{M}, t_0 \models \varphi(z) \text{ iff some } \delta_i(z) \text{ almost-holds in } \mathcal{M} \text{ at } t_0$$

Proof. Given an almost future $\varphi(z) = \bigvee \delta_i(z)$ as above, and given a left-limit moment t_0 in a linear structure $\mathcal{M}$:

Proof of $\Leftarrow$: t_0 is a left-limit, so there is an earlier moment $t < t_0$. Assume that some $\delta_i(z)$ almost-holds in $\mathcal{M}$ at t_0, then there is a $t' \in (t, t_0)$ such that $\mathcal{M}|_{\geq t'}, t_0 \models \delta_i(z)$, hence $\mathcal{M}|_{\geq t'}, t_0 \models \varphi(z)$, and as φ is almost future – $\mathcal{M}, t_0 \models \varphi(z)$ as well (Remark 3.2 (1)).

Proof of $\Rightarrow$: Assume that $\mathcal{M}, t_0 \models \varphi(z)$, then (by Remark 3.2 (2)) for every $t < t_0$ in $\mathcal{M}$: $\mathcal{M}|_{\geq t}, t_0 \models \varphi(z)$, hence for every $t < t_0$:

$$\mathcal{M}|_{\geq t}, t_0 \models \delta_i(z) \text{ for some index } i \tag{4}$$

Now, assume to the contrary that none of the disjuncts δ_i almost-holds in $\mathcal{M}$ at t_0. Then for each i there is a point – denote it by t_i – such that $t_i < t_0$ and for all $t' \in (t_i, t_0)$: $\delta_i(z)$ does not hold in $\mathcal{M}|_{\geq t'}$ at t_0. Let $\bar{t}$ denote the largest ('latest') t_i (we started off with a finite disjunction). t_0 is a left-limit, so there is a moment $t \in (\bar{t}, t_0)$. Thus for each i: $t_i \leq \bar{t} < t < t_0$, and therefore for each i $\delta_i(z)$ does not hold in $\mathcal{M}|_{\geq t}$ at t_0. This contradicts (4) above. Thus, we conclude that (at least) one of the disjuncts δ_i ***does*** almost-hold in $\mathcal{M}$ at t_0. □

The above lemma motivates us to seek a way to express in $TL(\mathsf{Until}, \mathsf{K}^-)$ the fact that "a formula almost-holds in $\mathcal{M}$ at t_0". The main technical lemma below shows that this is possible for $\overleftarrow{\exists\forall}$-formulas of both types, FM and non-FM:

Main Lemma 3.25. *For every FM-type or non-FM type $\overleftarrow{\exists\forall}$-formula $\psi^n(z)$ there is a $TL(\mathsf{Until},\mathsf{K}^-)$ formula F_{ψ^n} such that for every Dedekind complete structure $\mathcal{M}$ and every left-limit moment $t_0 \in \mathcal{M}$:*

$$\mathcal{M}, t_0 \models F_{\psi^n} \text{ iff } \psi^n(z) \text{ almost-holds in } \mathcal{M} \text{ at } t_0 \tag{5}$$

Proof. If $\psi^n(z)$ is a non-FM type, or a 'short' FM-type of the form $\psi^0 = (\langle True, \alpha_0 \wedge fm\rangle)$ let $F_{\psi^n} =_{def} False$, and then (5) above immediately follows by Remark 3.23 (3).

It remains to handle 'longer' FM-$\overleftarrow{\exists\forall}$-formulas:

Let $\psi^n(z) = (\langle \beta_n, \alpha_n \wedge fm\rangle, \langle \beta_{n-1}, \alpha_{n-1}\rangle, \ldots, \langle \beta_0, \alpha_0\rangle)$ be an FM-$\overleftarrow{\exists\forall}$-formula where $n \geq 1$, $\beta_n = True$ and α_n is a future formula (see Definition 3.20). Let A_i, B_i be $TL(\mathsf{Until},\mathsf{K}^-)$ formulas defining α_i, β_i for $0 \leq i \leq n$, and let $A_{n+1} =_{def} True$. Define $TL(\mathsf{Until},\mathsf{K}^-)$ formulas $G_0^{\psi^n}, G_1^{\psi^n}, \ldots, G_n^{\psi^n}, G_{n+1}^{\psi^n}$ and F_{ψ^n} as follows:

$$\begin{aligned}
&G_0^{\psi^n} =_{def} A_0 \\
&G_{j+1}^{\psi^n} =_{def} A_{j+1} \wedge (B_j \;\mathsf{Until}\; G_j^{\psi^n}) \qquad\qquad \text{for } j = 0, 1, \ldots, n \\
&F_{\psi^n} =_{def} A_0 \wedge \neg\mathsf{K}^-(\neg B_0) \wedge \bigwedge_{j=1}^{n+1} \mathsf{K}^-(G_j^{\psi^n})
\end{aligned}$$

Now let t_0 be a left-limit moment in a Dedekind complete structure $\mathcal{M}$. We show that F_{ψ^n} satisfies the required property (5).

Proof of property (5), $\Rightarrow$:

Assume that $\mathcal{M}, t_0 \models F_{\psi^n}$. Let $t < t_0$. To show that ψ^n(z) almost-holds in $\mathcal{M}$ at t_0 we must find a $t' \in (t, t_0)$ such that $\mathcal{M}|_{\geq t'}, t_0 \models \psi^n(z)$.

Below we will show how to find $n+1$ points: $t_1, \ldots, t_{n+1}$ in $\mathcal{M}$ such that (i) $t < t_{n+1} < \cdots < t_1 < t_0$ and (ii) for each $0 \leq i \leq n$: B_i holds in $\mathcal{M}$ along (t_{i+1}, t_i) and $\mathcal{M}, t_i \models G_i^{\psi^n}$ (and thus, in particular $\mathcal{M}, t_i \models A_i$). Having these points, let $t' = t_n$ (recall that $n \geq 1$), and then:

- For $0 \leq i < n$: As A_i, B_i are almost future, they hold in the corresponding points and intervals in the truncated structure $\mathcal{M}|_{\geq t'}$ as well (see Remark 3.2 (2)), and therefore α_i, β_i also hold in the same points and intervals in $\mathcal{M}|_{\geq t'}$.
- Additionally, we have $\mathcal{M}, t_n \models A_n$. But this is a future formula, and as $\mathcal{M}$ and $\mathcal{M}|_{\geq t'}$ coincide on $[t_n, \infty)$ it follows that $\mathcal{M}|_{\geq t'}, t_n \models A_n$ as well, and therefore $\mathcal{M}|_{\geq t'}, t_n \models \alpha_n$. And as t_n is a first-moment in $\mathcal{M}|_{\geq t'}$, clearly: $\mathcal{M}|_{\geq t'}, t_n \models \alpha_n \wedge fm$.
- Finally, β_n holds along $(-\infty, t_n)$ in $\mathcal{M}|_{\geq t'}$, as $t' = t_n$ is a first-moment in this structure.
- Thus we have shown that $\mathcal{M}|_{\geq t'}, t_0 \models \psi^n(z)$ as required.

It remains to show there are points t_i as promised above. First, as $\mathcal{M}, t_0 \models \neg\mathsf{K}^-(\neg B_0)$, we have an interval $(\bar{t}, t_0)$ where B_0 holds and $t < \bar{t} < t_0$. Second (recall that $n \geq 1$), as $\mathcal{M}, t_0 \models \mathsf{K}^-(G_{n+1}^{\psi^n})$, we have a $\dot{t} \in (\bar{t}, t_0)$ where $G_{n+1}^{\psi^n}$ holds, that is: $\mathcal{M}, \dot{t} \models (B_n \;\mathsf{Until}\; G_n^{\psi^n})$. For t_{n+1} we simply pick $\dot{t}$. Next, we construct t_n: We have $\mathcal{M}, t_{n+1} \models (B_n \;\mathsf{Until}\; G_n^{\psi^n})$, hence, $G_n^{\psi^n}$ holds at some $t'' > t_{n+1}$ and B_n holds along (t_{n+1}, t''). Now, if $t'' < t_0$ denote: $t_n = t''$. Otherwise, as $\mathcal{M}, t_0 \models \mathsf{K}^-(G_n^{\psi^n})$, there is a $t^* \in (t_{n+1}, t_0)$ where $G_n^{\psi^n}$ holds, and in this case denote: $t_n = t^*$. In any case, we have $t < \bar{t} < t_{n+1} < t_n < t_0$, B_n holds along (t_{n+1}, t_n) and $\mathcal{M}, t_n \models G_n^{\psi^n}$. Repeat the above arguments (induction, down-counting from t_n to t_1) to construct the rest of the t_i's. Finally, B_0 clearly holds along (t_1, t_0) and $\mathcal{M}, t_0 \models A_0$, so the points t_i indeed satisfy (i) and (ii) as required.

Proof of property (5), $\Leftarrow$**:**

Recall that t_0 is a left-limit moment in $\mathcal{M}$, and assume that $\psi^n(z)$ almost-holds in $\mathcal{M}$ at t_0. We prove that all three conjuncts of F_{ψ^n} hold in $\mathcal{M}$ at t_0:

- ***Third conjunct*** $\bigwedge_{j=1}^{n+1} \mathsf{K}^-(G_j^{\psi^n})$:

  Let $1 \leq j \leq n+1$ and let $\bar{t} < t_0$ in $\mathcal{M}$. To show that $\mathcal{M}, t_0 \models \mathsf{K}^-(G_j^{\psi^n})$, we must find a $t' \in (\bar{t}, t_0)$ such that $\mathcal{M}, t' \models G_j^{\psi^n}$. As t_0 is a left-limit, there is some $t^* \in (\bar{t}, t_0)$, and as $\psi^n(z)$ almost-holds in $\mathcal{M}$ at t_0, there is a $t \in (t^*, t_0)$ such that $\mathcal{M}|_{\geq t}, t_0 \models \psi^n(z)$. Thus (recall that $n \geq 1$), there are points $t_1, \ldots, t_n$ in $\mathcal{M}|_{\geq t}$ such that in $\mathcal{M}$: $\bar{t} < t^* < t \leq t_n < \cdots < t_1 < t_0$, and in $\mathcal{M}|_{\geq t}$:

  - $t \leq t_n < \cdots < t_1 < t_0$.
  - For $0 \leq i < n$: α_i (and therefore A_i) holds at t_i, and β_i (and therefore B_i) holds along (t_{i+1}, t_i). And as these are all almost future formulas (and as $t < t_i$), we conclude by Remark 3.2 (1) that A_i, B_i hold in the corresponding points and intervals in $\mathcal{M}$ as well.
  - $\alpha_n \wedge fm$ holds at t_n. Thus, t_n must be a first moment in $\mathcal{M}|_{\geq t}$, that is:

  $$t = t_n$$

  So, we have $\mathcal{M}|_{\geq t}, t_n \models \alpha_n$ (a future formula), and as $\mathcal{M}$ and $\mathcal{M}|_{\geq t}$ coincide on $[t_n, \infty)$, we conclude that $\mathcal{M}, t_n \models \alpha_n$ as well, and therefore $\mathcal{M}, t_n \models A_n$. And as $B_n = True$, B_n clearly holds in $\mathcal{M}$ along (t^*, t_n).

  Now, if $1 \leq j \leq n$ let $t' = t_j$ and if $j = n+1$ let $t' = t^*$. In either case we clearly have $\mathcal{M}, t' \models G_j^{\psi^n}$, and thus, by the definition of K^- it follows that:

  $$\mathcal{M}, t_0 \models \mathsf{K}^-(G_j^{\psi^n})$$

- ***First conjunct*** A_0:

  t_0 is a left-limit in $\mathcal{M}$, so indeed there is some $\bar{t} < t_0$ in $\mathcal{M}$, and we have already shown above that (given such $\bar{t}$) $\mathcal{M}, t_0 \models A_0$.

- ***Second conjunct*** $\neg\mathsf{K}^-(\neg B_0)$:

  Let $\bar{t} < t_0$. We have actually shown above (along the proof for the third conjunct) that there is a $t_1 \in (\bar{t}, t_0)$ such that B_0 holds in $\mathcal{M}$ all along the interval (t_1, t_0). By the definition of K^- it follows that:

  $$\mathcal{M}, t_0 \models \neg\mathsf{K}^-(\neg B_0)$$

Thus, $\mathcal{M}, t_0 \models F^{\psi^n}$.

□

3.4. Putting it all together

Lemma 3.25 renders the desired semantics-preserving translation over left-limit moments for almost future formulas:

Corollary 3.26 (Translation over left-limit points). *There is a translation $Trans^{ll}$ of FOMLO formulas into $TL(\mathsf{Until}, \mathsf{K}^-)$, such that for every almost future formula $\varphi(z)$:*

$$\varphi(z) \wedge ll(z) \equiv_{\mathcal{DC}} Trans^{ll}(\varphi) \wedge LL \tag{6}$$

Proof.

1. Given an almost future $\varphi(z)$, by Corollary 3.21 we have:

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_{i=1}^{k} \psi_i(z) \vee \bigvee_{i} \Big(\eta_i \wedge \bigwedge_{j} \chi_{ij}(z)\Big) \tag{7}$$

   where $k \geq 1$, ψ_i are all $\overleftarrow{\exists\forall}$-formulas of type FM or non-FM, η_i are all unbounded sentences and χ_{ij} are either unbounded sentences or $\overleftarrow{\exists\forall}$-formulas.

2. By Lemma 3.25, each ψ_i has a 'representative' F_{ψ_i} in $TL(\mathsf{Until}, \mathsf{K}^-)$ that satisfies property (5) of the lemma, or, in other words, that asserts that "$\psi_i(z)$ almost-holds in $\mathcal{M}$ at t_0". Define:

$$Trans^{ll}(\varphi) =_{def} \bigvee F_{\psi_i}$$

3. Notice that so far we haven't used the fact that φ is almost future: steps 1 and 2 above hold for any monadic $\varphi(z)$. Now verify that (6) above indeed holds: let t_0 be a left-limit moment in $\mathcal{M}$. By Lemma 3.24 (and this is the point where the "almost futureness" of φ is crucial), $\mathcal{M}, t_0 \models \varphi(z)$ iff at least one of the disjuncts of (7) above almost-holds in $\mathcal{M}$ at t_0. But recall that by Remark 3.23 (3) none of the η_i's can almost-hold anywhere, and by similar considerations none of the disjuncts of (7) that includes an 'unbounded' conjunct η_i can almost-hold anywhere. Thus $\mathcal{M}, t_0 \models \varphi(z)$ iff at least one of the ψ_i's almost-holds in $\mathcal{M}$ at t_0, in other words (by Lemma 3.25), iff there is a disjunct $\psi_i(z)$ such that $\mathcal{M}, t_0 \models F_{\psi_i}$, that is, iff $\mathcal{M}, t_0 \models Trans^{ll}(\varphi)$.

□

Now we are ready to complete the proof of our main result (Theorem 3.11): let $Trans^{fm}$, $Trans^{sc}$ be as in Lemma 3.12; and let $Trans^{ll}$ be as in Corollary 3.26 above. Given an almost future *FOMLO* formula $\varphi(z)$, denote:

$$F_\varphi =_{def} (FM \wedge Trans^{fm}(\varphi)) \vee (SC \wedge Trans^{sc}(\varphi)) \vee (LL \wedge Trans^{ll}(\varphi)) \tag{8}$$

Then F_φ is a $TL(\mathsf{Until}, \mathsf{K}^-)$ formula and, as every point in a Dedekind complete structure is (exclusively) either a first-moment or a successor or a left-limit (Remark 3.7), it follows by Lemma 3.12 and Corollary 3.26 that:

$$\varphi(z) \equiv_{\mathcal{DC}} F_\varphi$$

□

4. Further Results and Comments

We have shown expressive equivalence of $TL(\mathsf{Until}, \mathsf{K}^-)$ and almost future *FOMLO* over Dedekind complete time flows. The notion of past, future, almost future formulas is defined with respect to the class of all linear structures. One may as well consider similar notions relative to specific classes of structures. For example, a formula is a future formula over the class $\mathcal{DC}$ of all Dedekind complete time flows if any pair of Dedekind complete structures that coincide on the future of some point t agree on the formula at t. Clearly, every future formula over the class of all linear structures is also a future formula over $\mathcal{DC}$. The converse doesn't hold: The formula $P\ \mathsf{Until}'\ Q$ for example, which uses Stavi's modality to refer to *'gaps'* in time, is unsatisfiable over $\mathcal{DC}$, and therefore a future formula over $\mathcal{DC}$, but this is not a future formula over linear time domains.

We have stated our main result (Theorem 3.11) with respect to formulas which are almost future over all linear structures. The proof remains valid if, instead, we require "almost futureness" over the subclass $\mathcal{DC}$. Thus we have actually established a stronger version of the main result: every formula which is almost future over $\mathcal{DC}$ has a $TL(\mathsf{Until}, \mathsf{K}^-)$-equivalent over $\mathcal{DC}$. Theorem 3.11 then immediately follows, as every almost future formula is in particular almost future over Dedekind complete structures.

It is decidable whether a formula $\varphi(x)$ is almost future over Dedekind complete time. Indeed, let F_φ be obtained from any monadic formula $\varphi(x)$ as in (8) above, and let $\varphi'(x)$ be the standard translation of F_φ back into *FOMLO*. Then clearly the original $\varphi(x)$ is almost future over Dedekind complete time iff the formula $\varphi(x) \leftrightarrow \varphi'(x)$ is valid over Dedekind complete structures. Since the validity of a *FOMLO* formula over Dedekind complete structures is decidable [BG85], we conclude that it is decidable whether a formula is almost future over Dedekind complete structures.

In this paper we generalize the proof concerning real time domains which was presented in an extended abstract ([PR12]). The core translation we give here for left-limit points is the same construction presented there for real time domains. Still, the correctness proof for left-limit points had to be carefully adapted. Additionally, Proposition 3.18 (as already stated in [Ra14]) corrects a mistake in [PR12], where we have erroneously presented a decomposition of monadic formulas into $\exists\forall$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$.

Over linear structures in general, $\{\mathsf{Until}, \mathsf{K}^-\}$ is not expressive enough: It is not a basis for almost future formulas. Stavi generalized Kamp's theorem by enhancing $\{\mathsf{Until}, \mathsf{Since}\}$ to obtain a basis expressively equivalent to *FOMLO* over linear time [GHR94]. Unfortunately, $\{\mathsf{Until}, \mathsf{K}^-\}$ cannot be extended in a similar manner: no finite basis of almost future modalities is expressively equivalent to almost future *FOMLO* over linear time. A proof for this negative result was presented in [Ra15].

Table 1 summarizes which fragments of *FOMLO* can be accurately captured by an expressively equivalent Temporal Logic with a finite basis, depending on the kind of time flow in question:

	Time Flows	***Fragment of FOMLO***	***Expressively Equivalent Temporal Basis***
Kamp (1968)	Dedekind complete	*FOMLO*	$\{$ Until, Since $\}$
Stavi (1980)	Linear	*FOMLO*	$\{$ Until, Since, Until$'$, Since$'$ $\}$
[GPSS80]	Discrete ($\mathbb{N}, \mathbb{Z}, \dots$)	Future	$\{$ Until $\}$
[HR03]	Real	Future	No finite basis
Current paper	Dedekind complete	Almost future	$\{$ Until, K^- $\}$
[Ra15]	Linear	Almost future	No finite basis

Table 1: The Expressive Power of Temporal Logics

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