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Star free expressions over the reals

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Abstract

An interpretation of star free expressions over the reals is provided. The expressive power of star free expressions is compared to the expressive power of monadic first-order logic of order over the reals. It is proved that these formalisms have the same expressive power. This result provides a generalization of the classical McNaughton–Papert theorem (1971) from the finite orders to the order of the reals. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A fundamental theorem due to McNaughton and Papert [3] states that a regular language is definable by a star free expression if and only if it is definable in first order monadic logic of order. This theorem was extended to ω-regular languages in Ladner [2] and Thomas [9]. Moreover, more refined results were obtained in Thomas [10] and in Perrin and Pin [4], which show the correspondence between fragments of monadic first order logics and dot-depth hierarchy of star free expressions. The above results deal with discrete (time) linear orders.

In this paper an interpretation of star free expressions over the reals is provided. The expressive power of star free expressions is compared to the expressive power of monadic logic of order over the reals. It is proved that these formalisms have the same expressive power. This result is analogous to McNaughton-Papert theorem [3].

Our interest in star free expressions over the reals is motivated by Duration and Mean Value Calculi [11, 12]. These calculi are interval based formalisms for the specification of real time systems. They were successfully applied in a number of case studies

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$[\![\sigma]\!]_I$	$=$	$\{\eta : \eta(a) = \sigma$ for some $a \in I\}$
$[\![TRUE]\!]_I$	$=$	$\mathbf{R} \to \Sigma$
$[\![FALSE]\!]_I$	$=$	$\emptyset$
$[\![E_1; E_2]\!]_I$	$=$	$\{\eta : \eta \in [\![E_1]\!]_{I_1}$ and $\eta \in [\![E_2]\!]_{I_2}$ for a partition of I into subintervals $I_1, I_2\}$
$[\![E_1 \vee E_2]\!]_I$	$=$	Union of $[\![E_1]\!]_I$ and $[\![E_2]\!]_I$
$[\![E_1 \wedge E_2]\!]_I$	$=$	Intersection of $[\![E_1]\!]_I$ and $[\![E_2]\!]_I$
$[\![\neg E]\!]_I$	$=$	The complement of $[\![E]\!]_I$ with respect to $\mathbf{R} \to \Sigma$

Fig. 1. Definition of $[\![E]\!]_I$.

of software embedded systems [7] and were used to define the real time semantics of other languages. In [5] we show that there exist meaning preserving translations between the Propositional fragment of Mean Value Calculus and star free expressions. Hence, the expressive completeness of PMVC is obtained as a consequence of the expressive completeness of star-free expressions.

In this section we fix some notations and terminology and state our main result.

We use $\mathbf{R}$ for the set of real numbers. Recall that a nonempty subset I of reals is called an interval if $\forall c \forall a \in I \forall b \in I. a < c < b \to c \in I$. An interval I_1 precedes interval I_2 if $a \in I_1 \wedge b \in I_2 \to a < b$.

A partition of an interval I is an ordered pair of disjoint intervals I_1 and I_2 such that $I = I_1 \cup I_2$ and I_1 precedes I_2.

We will use standard notations for the intervals, e.g., for $a < b$ an open interval with endpoints a and b is denoted by (a, b).

Let Σ be a finite set. A Σ-predicate (over the reals) is a monadic function from the reals into Σ. We will use η to range over the Σ-predicates. A set of Σ-predicates is called a Σ-language.

It is clear that there exists a natural correspondence between n-tuples of boolean predicates over the reals and the $\{0, 1\}^n$-predicates.

Star free expressions over a finite set $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ are defined by the following grammar:

$$E ::= \sigma \mid TRUE \mid FALSE \mid E; E \mid E \vee E \mid E \wedge E \mid \neg E, \text{ where } \sigma \in \Sigma.$$

The Σ-language $[\![E]\!]_I$ specified by a star free expression E and an interval I is defined in Fig. 1.

A Σ-language L is *definable* by E if $L = [\![E]\!]_{\mathbf{R}}$.

The signature of the monadic language of order contains one binary predicate symbol $<$ and monadic predicate symbols. However, it will be more convenient for our purposes instead of dealing with several monadic predicate symbols to use one monadic Σ-predicate symbol X. Therefore, the atomic formulas of our language will be formulas *TRUE*, *FALSE*, $u < v$ and $X(u) = \sigma$, where u, v range over variables and $\sigma \in \Sigma$. The formulas are constructed from atomic formulas by the connectives $\wedge$, $\vee$, $\neg$ and the existential quantifier $\exists$.

Free and bound variables are defined as usual. We will use the notation $\phi\{u/v\}$ for the formula obtained from ϕ by replacing all free occurrences of v by u and renaming bound variables, if necessary. If all free variables of ϕ are among $\{t_1,\dots,t_n\}$, we write $\phi(t_1,\dots,t_n)$. Recall that a sentence is a formula without free variables.

The notion of satisfaction (in $\mathbf{R}$) is defined as usual. We write $\eta, a_1,\dots,a_n \models \phi(t_1,\dots,t_n)$ if $\phi(t_1,\dots,t_n)$ holds whenever X is interpreted as Σ-predicate η over the reals and the variables $t_1,\dots,t_n$ are interpreted as real numbers $a_1,\dots,a_n$.

A Σ-language L is *definable* by a sentence ϕ if $L = \{\eta : \eta \models \phi\}$.

We say that a star free expression E is equivalent to a sentence ϕ if E and ϕ define the same Σ-language, i.e., $\eta \in [\![E]\!]_{\mathbf{R}}$ iff $\eta \models \phi$.

The main result of this paper is

Theorem 1. *There exists a translation algorithm Tr from star free expressions into sentences of monadic first-order logic of order such that E is equivalent (over the reals) to $Tr(E)$. There exists a translation algorithm Tr' from sentences of monadic first-order logic of order to star free expressions such that ϕ is equivalent (over the reals) to $Tr'(\phi)$.*

This theorem follows from Theorem 9 (Section 3) and Theorem 14 (Section 4).

We have not analyzed the complexity of our translation algorithms which are clearly not optimal. The complexity of the satisfiability problem is non-elementary both for first-order monadic logic over the reals [8] and for star-free expressions [6]. We know [5] that there exists at least an exponential gap between succinctness of monadic logic and that of star-free expressions (i.e., there exists at least an exponential blow-up in every meaning preserving translation from monadic logic to star free expressions).

The rest of the paper is organized as follows. In section 2 we collect some immediate consequences of our definitions. Section 3 provides a translation from star free expressions to monadic logic. Section 4 presents the translation from monadic logic to star free expressions. A proposition due to Gabbay, Hodkinson and Reynolds [1] plays a central role in this translation. Finally, Section 5 states generalizations of our results to Dedekind closed linear orders. In particular, the McNaughton–Papert theorem is a special case of these generalizations.

2. Preliminaries

The standard syntactical extension of monadic logic by bounded quantifiers is given in this section. We also state some lemmas which are referred later.

Notation. The restriction of η to an interval I is denoted by $\eta|I$.

Definition 1. A Σ-language L is fictitious outside an interval I if $\eta|I = \eta'|I$ implies that $\eta \in L$ iff $\eta' \in L$.

The following lemma is immediate.

Lemma 2. 1. *$[\![E]\!]_I$ is fictitious outside the interval I.*
2. *Let f be a monotonic bijection on the reals that maps an interval I_1 onto interval I_2. Then $\eta \circ f \in [\![E]\!]_{I_1}$ iff $\eta \in [\![E]\!]_{I_2}$. In particular, $\eta \in [\![E]\!]_{\mathbf{R}}$ iff $\eta \circ f \in [\![E]\!]_{\mathbf{R}}$.*
3. *Let f be a monotonic bijection from a finite length interval $I = (a, b)$ onto the set of reals. If $\eta \circ f = (\eta' | I)$ then $\eta \in [\![E]\!]_{\mathbf{R}}$ iff $\eta' \in [\![E]\!]_I$.*

Lemma 3. *For every quantifier free formula $\phi(t)$ there exists a star free expression E such that $\eta \in [\![E]\!]_I$ if and only if there exists b such that I is a one point interval $\{b\}$ and $\eta, b \models \phi(t)$.*

Proof. Note that every quantifier free formula $\phi(t)$ is equivalent to one of the following formulas: *TRUE*, *FALSE* or $\bigvee_{\sigma \in \Sigma'} X(t) = \sigma$, where $\Sigma' \subseteq \Sigma$.

Let *POINT* be defined as $\neg(TRUE; TRUE)$. The lemma follows from the observations that (1) $\eta \in [\![POINT]\!]_I$ if I is one point interval. (2) $\eta \in [\![POINT \wedge (\bigvee_{\sigma \in \Sigma'} \sigma)]\!]_I$ iff $I = \{b\}$ and $\eta, b \models \bigvee_{\sigma \in \Sigma'} X(t) = \sigma$ for some $b \in \mathbf{R}$. □

It is convenient to extend the syntax of first order monadic logic of order by the bounded existential quantifiers $(\exists t)_{t_1}^{t_2}$. Semantically, $(\exists t)_{t_1}^{t_2} \phi$ is a shorthand for $\exists t.\ t_1 < t < t_2 \wedge \phi$. The variable t_1 (respectively t_2) is called the lower (respectively the upper) limit of the quantifier $(\exists t)_{t_1}^{t_2}$. Both t_1 and t_2 are free in $(\exists t)_{t_1}^{t_2} \phi$.

Let ϕ be a formula with only bounded quantifiers (without loss of generality we assume that each variable name is bound at most once in ϕ). A sequence $t_1, \ldots, t_n$ is called a *lower* (*upper*) sequence of ϕ if (1) t_1 is a bound variable of ϕ; (2) t_{i+1} is the lower (respectively, the upper) limit of the quantifier that binds t_i and (3) t_n is free in ϕ.

Example 1. *TRUE* and $X(t_1) = \sigma_1 \wedge t_1 < t_3$ do not have any lower and any upper sequences. The lower (respectively upper) sequences of $X(t_1) = \sigma_1 \wedge (\exists v)_u^w (X(v) = \sigma_2 \wedge \neg(\exists t_2)_{t_1}^v X(t_2) = \sigma_1)$ are t_2, t_1 and v, u (respectively t_2, v, w and v, w).

A formula is said to be *explicitly restricted* to $[t_1, t_2]$ if (1) all the quantifiers of the formula are bounded, (2) the set of its free variables is a subset of $\{t_1, t_2\}$ and (3) every lower sequence of the formula ends with t_1 and every upper sequence ends with t_2. We say that $\phi(t_1, t_2)$ is explicitly restricted to $(t_1, t_2]$ (respectively, $[t_1, t_2)$, or respectively (t_1, t_2)) if ϕ is explicitly restricted to $[t_1, t_2]$ and it does not contain an occurrence of $X(t_1)$ (respectively, $X(t_2)$, or, respectively, $X(t_1)$ and $X(t_2)$).

Example 2. It is clear that if all quantifiers of $\phi(t_1, t_2)$ are relativized to (t_1, t_2), i.e., have the form $(\exists v)_{t_1}^{t_2}$ then $\phi(t_1, t_2)$ is explicitly restricted to $[t_1, t_2]$. Note that according to our definition the formula $t_1 < t \wedge t < t_2$ is not explicitly restricted to $[t_1, t_2]$.

Notations. We denote by $Lang(\phi(t_1,\dots,t_n),a_1,\dots,a_n)$ the Σ-language $\{\eta : \eta,\ a_1,\dots,a_n \models \phi(t_1,\dots,t_n)\}$.

The following lemmas are straightforward.

Lemma 4. *If $\phi(t_1, t_2)$ is explicitly restricted to $[t_1, t_2]$ then $Lang(\phi(t_1, t_2), a_1, a_2)$ is fictitious outside $[a_1,a_2]$. Similarly, for the formulas explicitly restricted to $[t_1, t_2)$, $(t_1, t_2]$ or (t_1, t_2).*

Lemma 5. *Let $\phi(t_1, t_2)$ be a formula explicitly restricted to (t_1, t_2) and ψ be a sentence obtained from ϕ as follows*: (1) *Eliminate bounded quantifiers i.e., replace* "$(\exists v)_u^w\varphi$" *by* "$\exists v.u < v < w \wedge \varphi$"; (2) *Replace the sub-formulas $t_1 < u$ and $v < t_2$ by TRUE and* (3) *Replace the sub-formulas $u < t_1$ and $t_2 < u$ by FALSE. Let f be a monotonic bijection from a finite interval $I = (a, b)$ onto the set of reals. If $\eta \circ f = (\eta'|I)$ then $\eta \models \psi$ iff $\eta', a, b \models \phi(t_1, t_2)$.*

Lemma 6. *Assume that* (1) *every lower sequence of $\phi(t_1, t, t_2)$ ends at t_1 or at t;* (2) *every upper sequence of $\phi(t_1, t, t_2)$ ends at t_2 or at t and* (3) *there are no occurrences of $X(t_1)$ and $X(t_2)$ in $\phi(t_1, t, t_2)$. Then*

$$\forall abc.\ Lang(t_1 < t < t_2 \wedge \phi(t_1, t, t_2), a, c, b) \text{ is fictitious outside } (a, b)$$

3. From star free expressions to logic

In this section it will be proved that every star free expression is equivalent to a monadic sentence.

We say that a formula $\phi(t_1)$ is *equivalent* to a star free expression E *over one point interval* $I = \{a\}$ if $\eta \in [\![E]\!]_I$ iff $\eta,\ a \models \phi(t_1)$. Let $a < b$ be real numbers. We say that $\phi(t_1,t_2)$ is *equivalent* to E over an *interval I with endpoints a, b* if $\eta \in [\![E]\!]_I$ iff $\eta \in Lang(\phi(t_1, t_2)\, a, b)$.

In Fig. 2 five translations Tr_p, $Tr_{(\ )}$, $Tr_{(\ ]}$, $Tr_{[\ )}$, $Tr_{[\ )}$ from star free expressions into formulas are defined. In this definition $Tr \in \{Tr_p,\ Tr_{(\ )},\ Tr_{(\ ]},\ Tr_{[\ )},\ Tr_{[\ )}\}$. The translations of $E_1; E_2$ are based on all possible partitions of a finite interval into two subintervals. These partitions are summarized in Fig. 3. We use there notations ( ), [ ), ( ] and [ ] for the finite intervals on the reals, which are of the form (a, b), $[a, b)$, $(a, b]$ and $[a, b]$ for some $a < b$. Note that every finite length interval over the reals is either one point interval, or has two endpoints $a < b$ and is of the form $[a, b]$, $(a, b), [a, b)$, $(a,b]$.

It is easy to check that the following lemma holds.

Lemma 7. (Syntactical properties of the translations).

1. *Tr_p maps star free expressions to the quantifier free formulas with* (*at most*) *one variable t_1.*
2. *$Tr_{(\ )}$ maps star free expressions to the formulas explicitly restricted to $(t_1,\ t_2)$.*

$$
\begin{array}{lll}
Tr(C) & = & C,\ where\ C \in \{TRUE,\ FALSE\} \\
Tr(E_1\ op\ E_2) & = & Tr(E_1)\ op\ Tr(E_2),\ where\ op \in \{\wedge,\ \vee\} \\
Tr(\neg E) & = & \neg Tr(E) \\
Tr_p(\sigma) & = & X(t_1) = \sigma \\
Tr_{[\ ]}(\sigma) & = & X(t_1) = \sigma \vee X(t_2) = \sigma \vee (\exists t)_{t_1}^{t_2} X(t) = \sigma \\
Tr_{(\ ]}(\sigma) & = & X(t_2) = \sigma \vee (\exists t)_{t_1}^{t_2} X(t) = \sigma \\
Tr_{[\ )}(\sigma) & = & X(t_1) = \sigma \vee (\exists t)_{t_1}^{t_2} X(t) = \sigma \\
Tr_{(\ )}(\sigma) & = & (\exists t)_{t_1}^{t_2} X(t) = \sigma \\
Tr_p(E_1; E_2) & = & FALSE \\
Tr_{(\ )}(E_1; E_2) & = & (\exists t)_{t_1}^{t_2}.\ Tr_{(\,)}(E_1)\{t/t_2\} \wedge Tr_{[\,)}(E_2)\{t/t_1\} \\
 & & \vee (\exists t)_{t_1}^{t_2}.\ Tr_{(\ ]}(E_1)\{t/t_2\} \wedge Tr_{(\ )}(E_2)\{t/t_1\} \\
Tr_{(\ ]}(E_1; E_2) & = & (\exists t)_{t_1}^{t_2}.\ Tr_{(\ )}(E_1)\{t/t_2\} \wedge Tr_{[\ ]}(E_2)\{t/t_1\} \\
 & & \vee (\exists t)_{t_1}^{t_2}.\ Tr_{(\ ]}(E_1)\{t/t_2\} \wedge Tr_{(\ ]}(E_2)\{t/t_1\} \\
 & & \vee Tr_{(\ )}(E_1) \wedge Tr_p(E_2)\{t_2/t_1\} \\
Tr_{[\ )}(E_1; E_2) & = & (\exists t)_{t_1}^{t_2}.\ Tr_{[\ )}(E_1)\{t/t_2\} \wedge Tr_{[\ )}(E_2)\{t/t_1\} \\
 & & \vee (\exists t)_{t_1}^{t_2}.\ Tr_{[\ ]}(E_1)\{t/t_2\} \wedge Tr_{(\ )}(E_2)\{t/t_1\} \\
 & & \vee Tr_p(E_1) \wedge Tr_{(\ )}(E_2) \\
Tr_{[\ ]}(E_1; E_2) & = & (\exists t)_{t_1}^{t_2}.\ Tr_{[\ )}(E_1)\{t/t_2\} \wedge Tr_{[\ ]}(E_2)\{t/t_1\} \\
 & & \vee (\exists t)_{t_1}^{t_2}.\ Tr_{[\ ]}(E_1)\{t/t_2\} \wedge Tr_{(\ ]}(E_2)\{t/t_1\} \\
 & & \vee Tr_p(E_1) \wedge Tr_{(\ ]}(E_2) \\
 & & \vee Tr_{[\ )}(E_1) \wedge Tr_p(E_2)\{t_2/t_1\}
\end{array}
$$

Fig. 2. Translations.

Interval	Possible partitions into two subintervals			
Point	No partition			
( )	( ] and ( )	( ) and [ )		
( ]	( ] and ( ]	( ) and point	( ) and []	
[ )	[ ) and [ )	[ ] and ( )	point and ( )	
[ ]	[ ) and [ ]	[ ] and ( ]	[ ) and point	point and ( ]

Fig. 3. Partitions of finite length intervals over the reals.

3. $Tr_{[\ )}$ *maps star free expressions to the formulas explicitly restricted to* $[t_1,\ t_2)$.
4. $Tr_{(\ ]}$ *maps star free expressions to the formulas explicitly restricted to* $(t_1,\ t_2]$.
5. $Tr_{[\ ]}$ *maps star free expressions to the formulas explicitly restricted to* $[t_1,\ t_2]$.

The following proposition shows that our translations are correct.

Proposition 8. *Let* $a < b$ *be real numbers.*
1. $Tr_p(E)$ *is equivalent to* E *over all one point intervals.*
2. $Tr_{(\ )}(E)$ *is equivalent to* E *over the intervals of the form* $(a,\ b)$.

3. *Similarly*, $Tr_{(\ ]}(E)$ ($Tr_{[\ )}(E)$, $Tr_{[\ ]}(E)$)) *is equivalent to E over the intervals of the form* $(a, b]$ (*respectively* $[a, b)$, $[a, b]$).

Proof. The proof proceeds by a structural induction on the expressions. The base case is trivial. The only nontrivial inductive step is for the expressions of the form $E_1; E_2$. Below the proof is given for $Tr_{(\,)}$. For the other translations the proof is similar.

Assuming that Proposition 8 holds for expressions E_1 and E_2 we will derive that Proposition 8(2) holds for $E_1; E_2$.

Let $\phi_1(t_1, t_2) = Tr_{(\ )}(E_1)$ and let $\phi_1'(t_1, t_2) = Tr_{(\,]}(E_1)$. By the inductive assumption, for every $a < c$,

$$\eta, a, c \models \phi_1(t_1, t_2) \text{ if and only if } \eta \in [\![E_1]\!]_{(a,c)} \tag{1}$$

$$\eta, a, c \models \phi_1'(t_1, t_2) \text{ if and only if } \eta \in [\![E_1]\!]_{(a,c]} \tag{2}$$

Let $\phi_2(t_1, t_2) = Tr_{[\ )}(E_2)$ and let $\phi_2'(t_1, t_2) = Tr_{(\ )}(E_2)$. By the inductive assumption, for every $c < b$,

$$\eta, c, b \models \phi_2(t_1, t_2) \text{ if and only if } \eta \in [\![E_2]\!]_{[c,b)}, \tag{3}$$

$$\eta, c, b \models \phi_2'(t_1, t_2) \text{ if and only if } \eta \in [\![E_2]\!]_{(c,b)}. \tag{4}$$

Therefore, for $c \in (a, b)$,

$$\begin{gathered}\eta, a, c, b \models \phi_1(t_1, t_2)\{t/t_2\} \wedge \phi_2(t_1, t_2)\{t/t_1\} \\ \text{if and only if } \eta \in [\![E_1]\!]_{(a,c)} \text{ and } \eta \in [\![E_2]\!]_{[c,b)}\end{gathered} \tag{5}$$

(In (5) t_1, t, t_2 are interpreted as a, c, b.)

Therefore,

$$\begin{gathered}\eta, a, b \models (\exists t)_{t_1}^{t_2} \phi_1(t_1, t_2)\{t/t_2\} \wedge \phi_2(t_1, t_2)\{t/t_1\} \\ \text{if and only if there exists } c \in (a, b) \text{ such that} \\ \eta \in [\![E_1]\!]_{(a,c)} \text{ and } \eta \in [\![E_2]\!]_{[c,b)}\end{gathered} \tag{6}$$

Similarly,

$$\begin{gathered}\eta, a, b \models (\exists t)_{t_1}^{t_2} \phi_1'(t_1, t_2)\{t/t_2\} \wedge \phi_2'(t_1, t_2)\{t/t_1\} \\ \text{if and only if there exists } c \in (a, b) \text{ such that} \\ \eta \in [\![E_1]\!]_{(a,c]} \text{ and } \eta \in [\![E_2]\!]_{(c,b)}\end{gathered} \tag{7}$$

Recall that $\eta \in [\![E_1; E_2]\!]_{(a,b)}$ iff either there exists $c \in (a, b)$ such that $\eta \in [\![E_1]\!]_{(a,c)}$ and $\eta \in [\![E_2]\!]_{[c,b)}$ or there exists $c \in (a, b)$ such that $\eta \in [\![E_1]\!]_{(a,c]}$ and $\eta \in [\![E_2]\!]_{(c,b)}$. Hence, from (6) and (7) and the definition of $Tr_{(\ )}$ it follows that $\eta \in [\![E_1; E_2]\!]_{(a,b)}$ iff $\eta, a, b \models Tr_{(\ )}(E_1; E_2)$. This completes the inductive step for Proposition 8(2). The proof for 8(3) is similar to the above proof. The proof for 8(1) is straightforward. □

Theorem 9. *There exists a translation algorithm Tr from star free expressions into sentences such that E is equivalent to Tr(E).*

Proof. Let E be a star free expression and let $\phi(t_1, t_2) = Tr_{()}(E)$. Let the sentence ψ be obtained from $\phi(t_1, t_2)$ as in Lemma 5. To prove the theorem it is sufficient to show that

$$\eta \models \psi \text{ if and only if } \eta \in [\![E]\!]_{\mathbf{R}} \tag{8}$$

Let f be any monotonic bijection from the interval (0, 1) onto **R**. Assume that the restriction of η' onto interval (0, 1) is equal to $\eta \circ f$. By Lemma 5,

$$\eta \models \psi \text{ if and only if } \eta',\ 0,\ 1 \models \phi(t_1,\ t_2) \tag{9}$$

By Lemma 2(3),

$$\eta \in [\![E]\!]_{\mathbf{R}} \text{ if and only if } \eta' \in [\![E]\!]_{(0,\ 1)} \tag{10}$$

By Proposition 8(2),

$$\eta',\ 0,\ 1 \models \phi(t_1,\ t_2) \text{ if and only if } \eta' \in [\![E]\!]_{(0,\ 1)} \tag{11}$$

From (9)–(11) we obtain (8). □

4. From logic to star free expressions

In this section we show that every monadic sentence is equivalent to a star free expression. Our proof is based on the following proposition due to Gabbay et al. (see [1] Lemma 9.3.2).

Proposition 10. *There exists an algorithm that for every formula $\phi(t_1, t, t_2)$ constructs a formula $\psi(t_1, t, t_2)$ of the form $\bigvee_i(\psi^i_<(t_1) \wedge \psi^i_b(t_1) \wedge \psi^i_1(t_1, t) \wedge \psi^i_m(t) \wedge \psi^i_2(t, t_2) \wedge \psi^i_e(t_2) \wedge \psi^i_>(t_2))$ such that*

$$t_1 < t < t_2 \wedge \phi(t_1, t, t_2) \text{ is equivalent to } t_1 < t < t_2 \wedge \psi(t_1, t, t_2) \tag{12}$$

and

1. *ψ^i_b, ψ^i_m and ψ^i_e are quantifier free.*
2. *$\psi^i_1(t_1, t)$ and $\psi^i_2(t, t_2)$ are explicitly restricted to (t_1, t) and (t, t_2).*
3. *In $\psi^i_<(t_1)$ all quantifiers are of the form $\exists v.v < t_1 \wedge \chi$ and in $\psi^i_>(t_2)$ all quantifiers are of the form $\exists v.v > t_2 \wedge \chi$.*
4. *The quantifier depth of $\psi^i_1(t_1, t)$, $\psi^i_2(t, t_2)$ $\psi^i_<(t_1)$ and $\psi^i_>(t_2)$ is less than or equal to the quantifier depth of $\phi(t_1, t, t_2)$.*

Remark. In [1] it was shown that the equivalence (12) holds over arbitrary linear ordered set. Notice that the assertion that ψ is obtained from ϕ by an algorithm, and that the quantifier depth of the components ψ_1^i etc. are at most that of ϕ are corollaries of the proof of Lemma 9.3.2 in [1], not its statement.

Proposition 10 has the following

Corollary 11. *There exists an algorithm that for every formula $\phi(t_1, t, t_2)$ constructs a formula $\psi(t_1, t, t_2)$ of the form $\bigvee_i(\psi_1^i(t_1, t) \wedge \psi^i(t) \wedge \psi_2^i(t, t_2))$ such that if $Lang(t_1 < t < t_2 \wedge \phi(t_1, t, t_2), b, m, e)$ is fictitious outside (b, e) for every $b < e$ then*

$$t_1 < t < t_2 \wedge \phi(t_1, t, t_2) \text{ is equivalent to } t_1 < t < t_2 \wedge \psi(t_1, t, t_2) \tag{13}$$

1. *ψ^i are quantifier free.*
2. *$\psi_1^i(t_1, t)$ and $\psi_2^i(t, t_2)$ are explicitly restricted to (t_1, t) and (t, t_2) and their quantifier depth is less than or equal to the quantifier depth of $\phi(t_1, t, t_2)$.*

Proof. Let $\psi_<^i(t_1)$, $\psi_b^i(t_1)$ $\psi_1^i(t_1, t)$, $\psi_m^i(t)$ $\psi_2^i(t, t_2)$, $\psi_e^i(t_2)$ and $\psi_>^i(t_2)$ be as in Proposition 10. Let us choose $\sigma_1 \in \Sigma$ and let $\chi_1^i(t_1)$ (respectively $\chi_2^i(t_2)$) be the formula obtained from $\psi_<^i(t_1) \wedge \psi_b^i(t_1)$ (respectively from $\psi_e^i(t_2) \wedge \psi_>^i(t_2)$) through replacing all occurrences of $X(v) = \sigma_1$ by *TRUE* and all occurrences of $X(v) = \sigma$ (for $\sigma \neq \sigma_1$) by *FALSE*.

From the assumption that $Lang(t_1 < t < t_2 \wedge \phi(t_1, t, t_2), b, m, e)$ is fictitious outside (b, e) it follows that

$$\begin{aligned} &\eta, b, m, e \models t_1 < t < t_2 \wedge \phi(t_1, t, t_2) \text{ if only if} \\ &\eta, b, m, e \models t_1 < t < t_2 \wedge \bigvee_i (\chi_1^i(t_1) \wedge \psi_1^i(t_1, t) \wedge \psi_m^i(t) \wedge \psi_2^i(t, t_2) \wedge \chi_2^i(t_2)) \end{aligned} \tag{14}$$

Note that $\chi_1^i(t_1)$ is a formula that contains only the predicate symbol $<$ and therefore, it is equivalent over the reals to either *TRUE* or *FALSE*. (Such equivalence holds for any dense linear order without minimal and maximal elements. Moreover, it is decidable whether $\chi_1^i(t_1)$ is equivalent to *TRUE* or to *FALSE*.) Similarly, $\chi_2^i(t_2)$ is equivalent to either *TRUE* or *FALSE*. Therefore, the corollary is obtained from the above remark and (14) by defining

$$\psi^i(t) = \begin{cases} \psi_m^i(t) & \text{if both } \chi_1^i(t_1) \text{ and } \chi_2^i(t_2) \text{ are equivalent to } TRUE, \\ FALSE & \text{otherwise.} \end{cases} \quad \square$$

Proposition 12. *For every formula $\varphi(t_1, t_2)$ explicitly restricted to (t_1, t_2) there exists a star free expression E equivalent to $\varphi(t_1, t_2)$ over every open interval (a, b) for $a < b \in \mathbf{R}$.*

Proof. The proposition is proved by induction on the quantifier depth of $\varphi(t_1, t_2)$.

The base case is trivial because every quantifier free formula explicitly restricted to (t_1, t_2) is equivalent to either *TRUE* or to *FALSE*.

Observe that $\varphi_1 \vee \varphi_2$ is explicitly restricted to (t_1, t_2) iff both φ_1 and φ_2 are explicitly restricted to (t_1, t_2). Hence, if E_1 is equivalent to φ_1 and E_2 is equivalent to φ_2 then

$E_1 \vee E_2$ is equivalent to $\varphi_1 \vee \varphi_2$. Similar observations hold for conjunction and negation. Therefore, it is sufficient to carry the inductive step for the formulas $\varphi'(t_1, t_2)$ of the form $(\exists t)_{t_1}^{t_2}. \phi(t_1, t, t_2)$.

Recall that $(\exists t)_{t_1}^{t_2}. \phi(t_1, t, t_2)$ is defined as $\exists t.\ t_1 < t < t_2 \wedge \phi(t_1, t, t_2)$.

Note that since $\varphi'(t_1, t_2)$ is explicitly restricted to (t_1, t_2) it follows that (1) every low sequence of $\phi(t_1, t, t_2)$ ends at t_1 or t; (2) every upper sequence of of $\phi(t_1, t, t_2)$ ends at t_2 or t and (3) there is no occurrences of $X(t_1)$ and $X(t_2)$ in $\phi(t_1, t, t_2)$. Therefore, by Lemma 6

$$\forall abc.\ Lang(t_1 < t < t_2 \wedge \phi(t_1, t, t_2), a, c, b) \text{ is fictitious outside } (a, b)$$

Hence by Corollary 11,

$$\begin{gathered} \exists t.\, t_1 < t < t_2 \wedge \phi(t_1, t, t_2) \text{ is equivalent to} \\ \exists t.\, t_1 < t < t_2 \wedge (\bigvee_i (\psi_1^i(t_1, t) \wedge \psi^i(t) \wedge \psi_2^i(t, t_2)). \end{gathered} \tag{15}$$

Moreover, $\psi^i(t)$ are quantifier free, the quantifier depth of $\psi_1^i(t_1, t)$ and $\psi_2^i(t, t_2)$ is bounded by the quantifier depth of $\phi(t_1, t, t_2)$, and $\psi_1^i(t_1, t)$ (respectively $\psi_2^i(t, t_2)$) are explicitly restricted to (t_1, t) (respectively (t, t_2)).

Therefore, applying the inductive hypothesis we obtain that

$$\text{there are } E_1^i \text{ and } E_2^i \text{ which are equivalent to } \psi_1^i(t_1, t) \text{ and } \psi_2^i(t, t_2). \tag{16}$$

By Lemma 3

$$\begin{gathered} \text{there are } E^i \text{ such that } \eta \in [\![E^i]\!]_I \text{ if and only if there exists } c \\ \text{such that } I \text{ is a one point interval } \{c\} \text{ and } \eta, c \models \psi^i(t). \end{gathered} \tag{17}$$

From (15)–(17) we obtain that $(\exists t)_{t_1}^{t_2}. \phi(t_1, t, t_2)$ is equivalent to $\bigvee_i E_1^i; E^i; E_2^i$. This completes the inductive step. □

In Section 5 we will refer to the following:

Proposition 13. *For every formula $\varphi(t_1, t_2)$ explicitly restricted to $[t_1, t_2]$ there exists a star free expression E equivalent to $\varphi(t_1, t_2)$ over every closed interval $[a,b]$ for $a < b \in \mathbf{R}$. Similar results hold for the formulas explicitly restricted to $[t_1, t_2)$ and to $(t_1, t_2]$.*

Proof. The proposition follows from Proposition 12 and the observation that every formula explicitly restricted to $[t_1, t_2]$ is equivalent to a boolean combination of formulas explicitly restricted to (t_1, t_2) and formulas of the form $X(t_1) = \sigma$ and $X(t_2) = \sigma$. □

Finally, we show the second part of Theorem 1. Namely

Theorem 14. *There exists a translation algorithm Tr' from sentences to star free expressions such that ϕ is equivalent to $Tr'(\phi)$.*

Proof. Let ϕ be a monadic sentence. Without loss of generality we can assume that ϕ does not contain bounded quantifiers. Let t_1, t_2 be two variables that do not appear in ϕ and let $\varphi'(t_1, t_2)$ be a formula obtained from ϕ by relativizing all quantifiers to (t_1, t_2), i.e., replacing every quantifier "$\exists v.$" by bounded quantifier "$(\exists v)_{t_1}^{t_2}.$". It is clear that $\varphi'(t_1, t_2)$ is explicitly restricted to (t_1, t_2). Therefore, by Proposition 12 there exists a star free expression E such that for every $a < b \in \mathbf{R}$

$$\eta, a, b \models \varphi'(t_1, t_2) \text{ iff } \eta \in [\![E]\!]_{(a,b)} \tag{18}$$

We are going to show that E is the required expression, i.e.,

$$\eta \models \phi \text{ iff } \eta \in [\![E]\!]_{\mathbf{R}} \tag{19}$$

Indeed, applying the transformation from Lemma 5 we obtain from the formula $\varphi'(t_1, t_2)$ our sentence ϕ. Let f be a monotonic bijection from the interval $(0, 1)$ onto $\mathbf{R}$ and let η' be any predicate whose restriction to $(0, 1)$ is equal to $\eta \circ f$.

By Lemma 5

$$\eta \models \phi \text{ iff } \eta', 0, 1 \models \varphi'(t_1, t_2) \tag{20}$$

Moreover, by lemma 2(3),

$$\eta \in [\![E]\!]_{\mathbf{R}} \text{ iff } \eta' \in [\![E]\!]_{(0, 1)}. \tag{21}$$

The required conclusion (19) is obtained from (18), (20) and (21).

Finally, note that all the transformations in this proof are algorithmical. □

5. A generalization

We proved the equivalence of star free expressions and monadic first order logic of order over the reals.

The semantics of star free expressions can be defined for an arbitrary lineary ordered set A, namely the definition of the set of predicates over A specified by a star free expression E and a subinterval I of A is obtained from the definition in Fig. 1 by replacing the set of reals $\mathbf{R}$ by A; the notation $[\![E]\!]_I^A$ is used throughout this section for this set of predicates.

Our translation from star free expressions to monadic logic can be immediately generalized to any linear order $\langle A, <\rangle$ with the following properties: (1) Dedekind closure: if A_1 and A_2 are disjoint nonempty subsets of A such that $A = A_1 \cup A_2$ and $a_1 \in A_1 \wedge a_2 \in A_2 \rightarrow a_1 < a_2$, then there exists $c \in A$ such that $A_1 = \{a : a < c\}$ or $A_1 = \{a : a \leqslant c\}$. (2) Uniformity: for every $a < b \in A$ there exists a monotonic bijection from (a, b) onto A.

In the translation from logic to star free expressions we also used the following property (3) for every formula $\phi(t)$ that contains only the predicate symbol $<$ (no occurrence of monadic predicate symbol) $\forall ab. \in A.\ a \models \phi(t)$ iff $b \models \phi(t)$.

It is easy to see that property (2) implies property (3). Hence, for every linear order $\langle A, < \rangle$ with the properties (1) and (2) every monadic sentence is equivalent to a star free expression and every star free expression is equivalent to a monadic sentence.

We believe that the results can be generalized to any Dedekind closed order.

Let us point out that from the proof of Propositions 12 and 13 one can extract the following

Theorem 15 (From monadic logic to star free expressions over arbitrary linear orders).

1. *For every formula $\varphi(t_1, t_2)$ explicitly restricted to (t_1, t_2) there exists a star free expression E such that for every linear order A and every open subinterval (a_1, a_2) of A the formula $\varphi(t_1, t_2)$ is equivalent to E over (a_1, a_2), i.e., $[\![E]\!]^A_{(a_1,a_2)}$ coincides with the set of all monadic predicates over A which satisfy $\varphi(a_1, a_2)$.*
2. *For every formula $\varphi(t_1, t_2)$ explicitly restricted to $[t_1, t_2]$ there exists a star free expression E such that for every linear order A and every closed subinterval $[a_1, a_2]$ of A the formula $\varphi(t_1, t_2)$ is equivalent to E over $[a_1, a_2]$, i.e., $[\![E]\!]^A_{[a_1,a_2]}$ coincides with the set of all monadic predicates over A which satisfy $\varphi(a_1, a_2)$. Similar results hold for the formulas explicitly restricted to $[t_1, t_2)$ and $(t_1, t_2]$ respectively.*

The proof of Proposition 8 can be generalized to arbitrary Dedekind closed linear orders. Namely,

Theorem 16 (From star free expressions to monadic logic over Dedekind closed orders).

1. *For every star free expression E there exists a monadic formula $\varphi(t_1, t_2)$ explicitly restricted to (t_1, t_2) such that for every Dedekind closed linear order A and every open subinterval (a_1, a_2) of A the formula $\varphi(t_1, t_2)$ is equivalent to E over (a_1, a_2).*
2. *For every star free expression E there exists a monadic formula $\varphi(t_1, t_2)$ explicitly restricted to $[t_1, t_2]$ such that for every Dedekind closed linear order A and every closed subinterval $[a_1, a_2]$ of A the formula $\varphi(t_1, t_2)$ is equivalent to E over $[a_1, a_2]$.*
3. *Similar results hold for the intervals of the forms $[a_1, a_2)$ and of the forms $(a_1, a_2]$.*

Remark. Though the proof of Theorem 16 is similar to the proof of Proposition 8 there are some technical differences: for example, for a Dedekind closed linear order A, it might happen that for $a_1 < a_2 \in A$ the open interval $(a_1, a_2) = \{a : a_1 < a < a_2\}$ contains only one point. In order to treat such special cases one has to modify the translations given in Fig. 2.

Note that the McNaughton and Papert theorem [3] deals with finite linear orders. Clearly, such orders are Dedekind closed. Hence, McNaughton–Papert theorem is a consequence of Theorems 15 and 16.

We do not know whether the requirement of Dedekind closure is necessary in Theorem 16. In particular, it is an open question whether every star free expression is equivalent (over the order of rationals) to a monadic formula.

Finally, observe that the reals and the rationals have the same first-order monadic theory (i.e. a first-order monadic sentence is true on the reals if it is true on the rationals). However, there are star free expressions that are equivalent over the rationals but are not equivalent over the reals. The following example illustrates this observation.

Let O abbreviate the star-free expression $\neg POINT \wedge \neg(TRUE; POINT) \wedge \neg(POINT; TRUE)$, where $POINT = \neg(TRUE; TRUE)$ as in the proof of Lemma 3. So O expresses that an interval has no endpoints (is open). Now define the star-free expression $E = O{:}O$. This expresses that the linear order has a Dedekind cut given by two open intervals.

Now the set of rationals **Q** has such a cut, while **R** does not. Notice that the star-free expressions $O \to (O; O)$ and $TRUE$ are equivalent over the rationals but are not equivalent over the reals.

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References

[1] D. Gabbay, I. Hodkinson, M. Reynolds, Temporal Logic, Oxford Univ. Press, Oxford, 1994.
[2] R.E. Ladner, Application of model theoretical games to linear orders and finite automata theory, Inform. and Control 9 (1977) 521–530.
[3] R. McNaughton, S. Papert. Counter-free Automata, MIT Press, Cambridge, MA, 1971.
[4] D. Perrin, J.E. Pin, First order logic and star free sets, J. Comput. System Sci. 32 (1986) 393–406.
[5] A. Rabinovich, On expressive completeness of duration and mean value calculi, in preparation. The Extended Abstract, in the Electronic Notes in Theoretical Computer Science, vol. 7, 1997.
[6] A. Rabinovich, Non-elementary lower bound for propositional duration calculus, Inform. Process. Lett. 66 (1998) 7–11.
[7] A. Ravn, H. Richel, K. Hansen, Specifying and verifying requirement of real time systems, IEEE Trans. Software Eng., 1993.
[8] L. Stockmeyer, The complexity of decision problems in automata and logic, Ph.D. Thesis, MIT, 1974.
[9] W. Thomas, Star Free regular sets of ω-sequences, Inform. and Control 42 (1979) 148–156.
[10] W. Thomas, Classifying regular events in symbolic logic, J. Comput. System Sci. 25 (1982) 360–376.
[11] Zhou Chaochen, C.A.R. Hoare, A.P. Ravn. A calculus of duration, Inform. Process. Lett. 40 (5) (1991) 269–279.
[12] Zhou Chaochen, Li Xiaoshan, A mean value calculus of duration, in: A classical Mind: Essays in Honor of C.A.R. Hoare, Prentice-Hall, Englewood Cliffs, NJ, pp. 431–451.