Fundamenta Informaticae 62 (2004) 1–28
IOS Press

Logics for Real Time: Decidability and Complexity

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Abstract. Over the last fifteen years formalisms for reasoning about metric properties of computations were suggested and discussed. First as extensions of temporal logic, ignoring the framework of classical predicate logic, and then, with the authors' work, within the framework of monadic logic of order. Here we survey our work on metric logic comparing it to the previous work in the field. We define a quantitative temporal logic that is based on a simple modality within the framework of monadic predicate Logic. Its canonical model is the real line (and not an ω-sequence of some type). It can be interpreted either by behaviors with finite variability or by unrestricted behaviors. For finite variability models it is as expressive as any logic suggested in the literature. For unrestricted behaviors our treatment is new. In both cases we prove decidability and complexity bounds using general theorems from logic (and not from automata theory).

1. Introduction

. Temporal Logic (TL) is a convenient framework for reasoning about the evolving of a system in time and it enjoyed extensive research during the last 20 years. A celebrated theorem by Kamp [Kamp68] proves that the temporal logic based on the two modalities "Since" and "Until" has the same expressive power as the monadic first order predicate logic, whether time is taken as the set of natural numbers *(discrete time*) or as the set of non negative real numbers (*continuous time*). For discrete time there is also a tight connection between second order predicate logic and automata theory [Bü60], associating with every formula an automaton that accepts exactly the runs that satisfy the formula. This associates in particular an automaton with every first order logical formula. For classical logic, constructing the

automaton that corresponds to a first order formula is neither simpler nor more direct than for second order formulae. In contrast, for temporal logic the construction of the automaton is more straightforward. This may have contributed to the popularity of temporal logic for the treatment of discrete time.

When it came to continuous time most of the researchers decided to abandon the natural model of the real line in favor of discrete models in which the elements of the sequence were decorated by some information concerning the real line. This may have been an attempt to pursue the connection with automata theory, as automata were traditionally associated with sequences. We believe that this decision had far reaching consequences that complicated the subsequent research enormously. Firstly, the theory could only deal with systems that satisfied a finite variability condition. Admittedly, computer science maybe interested mainly in finite variability signals, but the natural way to build a language is to create a language that treats all signals and then tune it up to suit any restricted class of signals. Secondly, in some of the approaches it was not even clear which finite variability model was intended; we may choose if the "timed sequences" encode only discrete signals (which are 0 except possibly for an unbounded ω-sequence of points), or if they encode only right continuous signals (or only left continuous signals). Any other collection of signals has no natural encoding as timed sequences. Furthermore: The encoding is not one to one and we need to address the question if two sequences representing the same signal actually satisfy the same formulas (in some temporal languages they don't). Thirdly and most important, the equivalence between first order classical logic and temporal logic that held for the real line by Kamp's theorem was lost. It followed that the choice of which temporal logic to use became an arbitrary choice which was not backed up anymore by the knowledge that the classical logic has the same expressive power.

The loss of the classical logic yardstick handicapped the research of metric logics. The combination of timed sequences and temporal logic was a poor setting for introducing metric into the discussion. When dealing with the real line in predicate logic the addition of the $+$ function is just a trivial change in the "signature of the logic". The resulting logic is rich and it is the natural place to look for sublogics that express different properties of the metric. This obvious approach has no counterpart when the model is a sequences and the logic is a temporal logic. Thus a host of different formalisms was introduced which combined with the different sequential models produced an astonishing variety of pairs "model-logic". In spite of the efforts invested over the years the research did not converge towards an accepted sequential model and an appropriate metric temporal logic.

In this paper we expand our research from [HR99] and [HR99a]. Our underlining model is the ordered real line itself with unary predicates corresponding to properties that may or may not hold at each point. In section 2 we describe the classical background; we define the Predicate Monadic Logic of Order. Temporal logics are defined within this logic, and in particular we recall the temporal logic $TL(Since, Until)$ which is based on the two modalities "Since" and "Until". We recall Kamp's theorem that proves the expressive completeness of $TL(Since, Until)$. We recall the known decidability results for these logics which are not known well enough in the computer science community. We make the distinction between the class of all models, that describe an arbitrary behavior in time and the class of models with finite variability, that corresponds to behaviors that make only a finite number of changes in any finite amount of time. In section 3 we add to the logics very simple metric constructs. These are the constructs "φ will happen within one unit of time", and "φ happened within the last unit of time". These constructs are added to predicate logic as a sublanguage of the language with the $+1$ function, and to temporal logic through explicit new temporal connectives. We show that with the added expressive power the temporal logic and the classical logic are still equivalent in expressive power. In section 4 we show

that these simple metric logics are quite expressive. We show that they can express seemingly stronger alternative connectives, in particular all the connectives that we found in the literature. In sections 5 and 6 we reduce the metric logics, as far as satisfiability is concerned to the simple non metric case. Thus the metric logics inherit from the non metric logics the decidability and complexity results, which is the content of sections 7 and 8. We obtain in this way decidability and complexity results that are new for the class of general models. For finite variability models this yields new, automata free proof of decidability. The analysis starts in section 5 with the introduction of Timer Formulas. Their building stones include basic formulas like $Timer_1(X, Y)$ which says that Y holds at a point t iff X has been true at every moment of the last unit of time. The main technical theorem eliminates the metric entirely from timer formulas by reducing effectively each timer formula φ to a non metric formula φ' (i.e., formula in the monadic logic of order) such that φ is satisfiable iff φ' is satisfiable. This yields decidability results for classes of timer formulas that reduce to a decidable class of non metric formulas. To carry the analysis further we say that a formula has a timer normal form if it is equivalent to a timer formula. Section 6 completes the picture: all the formulas in the metric logic have a timer normal form which reduces to first order non metric monadic logic of order. Hence the problem of satisfiability for all the formulas in the metric logic that we chose is decidable whether we are interested in all the models or only in models with finite variability. For finite variability models this result (stated in different models) was obtained in [AH92]. A more careful analysis shows in section 8 that the reductions from temporal metric logic to timer normal form can be economical enough to retain the classical complexity results for finite variability (and in fact even improve upon them), again without using automata theory along the way (we emphasize the independence from automata theory as we think that the belief in the necessity of automata theory was a main reason for rejecting the classical model).

The method of reduction to the non metric case through timer normal form formulas can be extended to logics stronger than the metric logic that we discussed upto section 8; in particular second order formulas can be involved cautiously. This is the subject of section 9. We define a strong metric predicate logic which we call *Q2MLO* and which is still reducible to timer normal form and which is still decidable. In section 10 we investigate a host of strong temporal logics that fall under the scope of the logics discussed in section 9 and are therefore decidable. We find it hard to imagine a decidable temporal logic that does not have its modalities defined in the style of section 9. Finally section 11 compares our work to the previous work done in the field. Section 11.1 and 11.2 do it favorably; describing it briefly and trying to give credit where deserved. Section 11.3 is more critical and tries to show that the use of the sequence models without the uniform frame of reference of classical logic leads to difficulties.

To keep the size and the difficulty of the paper within reasonable limits we omitted the full proofs and left only hints or sketches that indicate the main ideas. Full proofs which are quite technical will have to wait for detailed papers dedicated to the different subjects covered.

2. Monadic Logic of Order and Temporal Logics

2.1. MLO - Monadic Logic of Order

The most natural language to discuss systems that evolve in time is classical predicate logic. The language has a name for the (discrete or continuous) order relation of the time line, and a supply of unary predicate each to denote a property which the system may or may not have at any point in time. This leads to the following definition of the syntax and the semantics of the pure (non metric) monadic predicate

logic:

The syntax of the monadic predicate logic of order - MLO has in its vocabulary *individual* (first order) variables $t_0, t_1, \ldots$, monadic *predicate* names $X_0, X_1, \ldots$, and one binary relation $<$ (the order). **Atomic formulas** are of the form $X(t), t_1 < t_2$ and $t_1 = t_2$. **Well formed formulas** of the monadic logic MLO are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$. **Second order** Monadic Logic of Order formulas are obtained using also second order quantifiers $\exists X$ and $\forall X$.

Warning on Terminology: In this work we are mainly interested in first-order monadic logic of order, and unless second order is explicitly mentioned the name MLO is reserved for first order logic.

A structure for MLO is a pair $M = \langle A, < \rangle$ where $<$ is a linear order over A. To be exact a structure $M = \langle A, <, P_1, \ldots, P_n \rangle$ may have some predetermined fixed one-place predicates (sets) $P_1, \cdots, P_n$, but we shall use the simple notation $\langle A, < \rangle$ when the particular special predicates are not essential to the discussion.

The important examples for us are $\langle R^+, < \rangle$ and $\langle N, < \rangle$; the non-negative real line and the non-negative integers.

We shall not repeat the standard inductive definition saying when a formula is satisfied. We recall the notation:

$$M, \tau_1, \ldots, \tau_k; P_1, \ldots, P_m \models_{MLO} \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$$

which we also abbreviate to $M \models \varphi[\tau_1, \ldots, \tau_k; P_1, \ldots, P_m]$ or even to $M \models \varphi[\overline{\tau}, \overline{P}]$, where M is a structure, $\tau_1, \ldots, \tau_k$ are elements of M, $P_1 \cdots, P_m$ are unary predicates (i.e sets) in M, φ is a formula and $t_1, \ldots, t_k; X_1, \ldots, X_m$ include all the variables of φ.

The non negative real line $\langle R^+, < \rangle$ and the non negative integers $\langle N, < \rangle$ respectively are called the **continuous canonical model** and the **discrete canonical model**.

Finite Variability: A function η from a bounded interval I (of reals or rationals) into a set Σ has *finite variability* if there exists a partition of I into a finite sequence $I_1, \ldots, I_k$ of subintervals such that η is constant on every subinterval I_j $(j = 1, \ldots, k)$. A function η from reals or rationals has finite variability if the restriction of η on every bounded interval has finite variability.

From the above definition it follows that a function η from a bounded subinterval $(a,\ b)$ of the reals into a set Σ has *finite variability* if there exists a finite increasing sequence $a = a_0 < a_1 \cdots < a_n = b$ such that η is constant on every interval (a_i, a_{i+1}). A function η from the non negative real line R^+ into a set Σ has *finite variability* if there exists an increasing unbounded ω-sequence $0 = a_0 < a_1 \cdots < a_n < \ldots$ such that η is constant on every interval (a_i, a_{i+1}).

In the computer science literature it is often assumed that the system has finite variability; i.e, that every monadic predicate involved has finite variability (we identify predicates with their characteristic functions).

The (first or second order) *Finite Variability Monadic Logic of Order* is syntactically the logic introduced before with the additional assumption concerning the semantics that all the predicate names and second-order variables range over finite variability predicates.

Remark 2.1. (About Finite Variability)

1. The distinction between the class of models with finite variability and the class of unrestricted models is significant also for first order monadic logic. It is easy to express in first order language the fact that any non-empty interval $(a,\ b)$ has a point where the predicate X holds and a point

where $\neg X$ holds. This formula is satisfiable in the class of unrestricted models but not in the class of finite variability models.

2. In the real line model, finite variability can be defined by a *first order* formula: The predicate X has finite variability iff it satisfies the formula

$$\forall t \exists t_1[(t < t_1) \wedge \forall t_2(t < t_2 < t_1) \rightarrow (X(t_2) \leftrightarrow X(t_1)] \\ \bigwedge \\ \forall t > 0 \exists t_1[(t_1 < t) \wedge \forall t_2(t_1 < t_2 < t) \rightarrow (X(t_2) \leftrightarrow X(t_1)]$$

($t > 0$ can be written as $\exists s(s < t)$).

3. In contrast, over the rational line finite variability can be defined by a second-order monadic formula but not by a first order one.

2.2. Decidability

We conclude the discussion of the monadic logic of order with a list of decidability results.

Theorem 2.2.
1. The satisfiability of monadic first order logic of order over the reals is decidable [BG85].
2. The satisfiability of monadic second-order logic of order over the reals is undecidable [She75]
3. The satisfiability of monadic second order logic over the reals with finite variability interpretation is decidable [Rab98].
4. The satisfiability of monadic second order logic over the rationals is decidable [Ra69].

2.3. Temporal Logics

Temporal logics evolved in philosophical logic and were enthusiastically embraced by a large body of computer scientists. It uses logical constructs called "modalities" to create a language that is free from variables and quantifiers. Every modality that was seriously considered corresponds to a formula in relational logic, and it is obvious that the temporal logic can be thought of as syntactical sugar for a fragment of classical logic. In its purest form the use of sugaring is harmless since Kamp proved that the expressive power of the temporal logic equals that of the monadic logic of order (concerning formulas with at most one free variable).

Here is the general logical framework to define temporal logics:

The syntax of the Temporal Logic $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)})$ has in its vocabulary *monadic predicate variables* $X_1, X_2, \ldots$ and a sequence of *modality names* with prescribed arity, $O_1^{(k_1)}, \ldots, O_n^{(k_n)}$ (the arity notation is usually omitted). The formulas of this temporal logic are given by the grammar:

$$\varphi \ ::= \ X \mid \neg\varphi \mid \varphi \wedge \varphi \mid O^{(k)}(\varphi_1, \cdots, \varphi_k)$$

When the particular modality names are unimportant or they are clear from the context we omit them and write TL instead of $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)})$.

Structures for TL are again linear orders with monadic predicates $M = \langle A, <, P_1, P_2, \ldots, P_n, \ldots \rangle$, where the predicate P_i is assigned to a variable X_i. Every modality $O^{(k)}$ is interpreted in every structure M as an operator $O^{(k)}_M : [\mathbb{P}(A)]^k \to \mathbb{P}(A)$ which assigns "the set of points where $O^{(k)}[S_1, \ldots, S_k]$ holds" to the k-tuple $\langle S_1, \ldots, S_k \rangle \in \mathbb{P}(A)^k$. (Here $\mathbb{P}$ is the power set notation, and $\mathbb{P}(A)$ denotes the set of all subsets of A.) Once every modality corresponds to an operator the semantics is defined by structural induction:

- for atomic formulas

$$\langle M, \tau \rangle \models_{TL} X \quad \text{iff} \quad \tau \in P, \text{ where the monadic predicate } P \text{ is assigned to } X.$$

- for Boolean combinations the definition is the usual one.
- for $O^{(k)}(\varphi_1, \cdots, \varphi_k)$

$$\langle M, \tau \rangle \models_{TL} O^{(k)}(\varphi_1, \cdots, \varphi_k) \quad \text{iff} \quad \tau \in O^{(k)}_M(A_{\varphi_1}, \cdots, A_{\varphi_k})$$

where $A_\varphi = \{ \tau : \langle M, \tau \rangle \models_{TL} \varphi \}$ (we suppressed predicate parameters that may occur in the formulas).

Actually we are interested in a more restricted case; for the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets A_{φ_i} of points satisfying φ_i and the set of points $O[A_{\varphi_1}, \ldots, A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

Truth Tables: A formula $\overline{O}(t_0, X_1, \ldots X_k)$ in the predicate logic L is a *Truth Table* for the modality $O^{(k)}$ if for every structure M

$$O_M(P_1, \ldots, P_k) = \{\tau : M \models_{MLO} \overline{O}[\tau, P_1, \ldots, P_k]\} .$$

The following are examples of modalities introduced through truth tables. The modalities *until* and *since* which are introduced here are the most commonly chosen modalities in temporal logic for computer science.

Example 2.3. (Tables for Modalities)

- The modality $\Diamond X$, " eventually X", is defined by $\varphi(t_0, X) \equiv \exists t > t_0 \; X(t)$.
- The modality $\overleftarrow{\Diamond} X$, "$X$ has happened before", is defined by $\varphi(t_0, X) \equiv \exists t < t_0 \; X(t)$.
- The modality $X \; \mathbf{U} \; Y$, "X *until* Y", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1 (t_0 < t_1 \wedge Y(t_1) \wedge \forall t (t_0 < t < t_1 \to X(t))).$$

- The modality $X \; \mathbf{S} \; Y$, "X *since* Y ", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1 (t_0 > t_1 \wedge Y(t_1) \wedge \forall t (t_1 < t < t_0 \to X(t))).$$

- $Always(X)$ is the modality that states that X always holds. Its truth table is: $(t_0 = t_0) \wedge \forall t. \; X(t)$.

The following observation justifies our view that temporal logics can be thought of as an alternative approach (sugaring) to fragments of the predicate logic in which they are defined:

Proposition 2.4. If every modality in the temporal logic TL has a truth table in the logic MLO then to every formula $\varphi(X_1, \ldots, X_n)$ of TL there corresponds effectively (and naturally) a formula $\overline{\varphi}(t_0, X_1, \ldots X_n)$ of MLO such that for every $M, \tau \in M$ and predicates $P_1, \ldots, P_n$

$$\langle M, \tau, P_1, \ldots, P_n \rangle \models_{TL} \varphi \quad \text{iff} \quad \langle M, \tau, P_1, \ldots, P_n \rangle \models_{MLO} \overline{\varphi}\,.$$

In particular by example 2.3, the temporal logic $TL(\,\mathbf{U}\,,\,\mathbf{S}\,)$ with the modalities "until" and "since" describes a fragment of (first-order) MLO.

2.4. Expressive Completeness of $TL(until, since)$

The discussion above brings up the question which simple collection of truth tables and modalities suffices to represent all of the monadic logic. Formally:

Definition 2.5. (Expressive Completeness)
Let L be a fragment of first-order logic and let C be a class of structures. A temporal logic TL' is expressively complete for L over C if for every formula $\psi(t_0, X_1, ..., X_n) \in L$ with single free first order variable t_0, there is a formula $\varphi \in TL'$ which is equivalent to ψ over C, and for every formula $\varphi \in TL'$, there is a formula $\psi(t_0, X_1, ..., X_n) \in L$ with single free first order variable t_0 which is equivalent to φ over C.

We could of course take all the formulas with one free variable of MLO as truth tables for modalities. This would yield an expressively complete, extremely uninteresting temporal logic. A much smaller collection of modalities suffices, as can be seen from the following beautiful theorem:

Theorem 2.6.
1. ([Kamp68], reproved in [GPSS80]): The temporal logic $TL(until, since)$ is expressively complete for (first-order) MLO over the two canonical structures: the real line and the natural numbers.
2. [GHR94] There is a pair of modalities $X until_s Y$ and $X since_s Y$ ("Stavi's modalities") which together with the *since* and *until* is expressively complete for (first-order) MLO over the class of all linear orders.

Hence from now on when we say simply "temporal logic" or "pure temporal logic" we mean $TL(until, since)$.

3. Quantitative Temporal Logic and Quantitative Monadic Logic of Order

The logics MLO and $TL(Until, Since)$ are not suitable to deal with quantitative statements like "X will occur within one unit of time". This can be easily remedied in predicate logic by using the function $t + 1$:

3.1. Extending MLO with the $+1$ Function

Definition 3.1. MLO^+, the monadic logic of order with a $+1$ function, is the first order logic built from the binary relation $<$, monadic predicate names and one unary function symbol which we denote either by $S(t)$ or by $t+1$. The *standard (canonical) model* for this logic is the non-negative real line with the usual $+1$ function.

General structures for MLO^+ are ordered structures $M = \langle A, <, S\rangle$ with the binary order predicate $<$ and with a unary function S that satisfies some natural requirements like strict monotonicity and being archimedean. To be exact a structure $M = \langle A, <, S, P_1, \cdots, P_n\rangle$ may have some predetermined fixed one place predicates (sets) $P_1, \cdots, P_n$, but we shall use the simple notation $\langle A, <, S\rangle$ when the particular special predicates are not essential to the discussion.

We will discuss only the continuous time **canonical model**: $\langle R^+, <, S\rangle$, the non negative reals with the function $S(t) = t+1$. (Over the discrete time canonical model the successor function is easily expressible from the order predicate.) We use the standard notation: 0 for the minimal element and n for $0+1+\cdots+1$ (n times).

MLO^+ is a strong language and everything that we do is inside MLO^+. But it is too strong: the problem of validity and satisfiability in the standard model is undecidable for MLO^+ (there is a natural encoding of Turing computations into the language). We shall, therefore, start at the other end: introduce the simplest quantitative modalities to TL and check what the corresponding fragment of MLO^+ is. We then check if the result is expressive enough and if it is decidable.

3.2. A Minimal Metric Extension of $TL(until, since)$

The most basic modalities in temporal logic are the modalities $\Diamond X$, "X will occur in the future" and $\overleftarrow{\Diamond} X$, "$X$ has occurred in the past". The most basic metric modalities are the modalities claiming that X will occur in the next unit of time or that X has occurred within the last unit of time.

Definition 3.2. (Quantitative Temporal Logic)
For any set of temporal modalities Γ which is expressively complete for the first order monadic logic of order we denote by $QTL(\Gamma)$ the temporal logic constructed from Γ *and* two new modalities $\Diamond_1 X$ and $\overleftarrow{\Diamond}_1 X$ defined by the tables with free variable t_0:

$$\Diamond_1 X: \qquad \exists t((t_0 < t < t_0 + 1) \wedge X(t)) \tag{3}$$

$$\overleftarrow{\Diamond}_1 X: \qquad \exists t((t < t_0 < t + 1) \wedge X(t))\,. \tag{4}$$

We shall denote by QTL the quantitative logic $QTL(Until, Since)$. We also introduce the derived modalities:

$$\Box_1 X = \neg\Diamond_1\neg X \qquad\qquad \overleftarrow{\Box}_1 X = \neg\overleftarrow{\Diamond}_1\neg X$$

Next we intend to identify the fragment of MLO^+ that corresponds to QTL. This fragment will use the function $t+1$ only in a very restricted form as indicated in (3) and (4). We introduce some syntactical sugar to MLO^+, the "bounded quantifiers" $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$. If φ is a formula of MLO^+ then we use the shorthand:

$$(\exists t)^{<t_0+1}_{>t_0}\varphi \equiv \exists t(t_0 < t < t_0 + 1 \wedge \varphi(t)) \tag{5}$$

We do not have substraction in the language and we define:

$$(\exists t)^{<t_0}_{>t_0-1}\varphi \equiv \exists t((t < t_0 < t+1) \wedge \varphi(t)) \tag{6}$$

Note that for points t_0 smaller than 1 it just means that there is some previous t satisfying φ.

We introduce the abbreviations $(\forall t)^{<t_0+1}_{>t_0}$ and $(\forall t)^{<t_0}_{>t_0-1}$ in the standard way.

Definition 3.3. Quantitative Monadic Logic of Order (*QMLO*) is the fragment of MLO^+ which is built from the atomic formulas $t_1 < t_2, t_1 = t_2, X(t)$ (t, t_1, t_2 variables) using Boolean connectives, first order quantifiers and the following rule: if $\varphi(t)$ is a formula of *QMLO with t its only first order free variable* then $(\exists t)^{<t_0+1}_{>t_0}\varphi$ and $(\exists t)^{<t_0}_{>t_0-1}\varphi$ are formulas of *QMLO* .

The following observation characterizes the expressive power of QTL.

Theorem 3.4. (Expressive completeness of QTL for *QMLO*)
Let Γ be an expressive complete set of modalities (over the reals) for first-order MLO.

1. For every formula $\psi(X_1, \ldots, X_l)$ over the modalities $\Gamma \cup \{\Diamond_1, \overleftarrow{\Diamond}_1\}$ there is a formula $\varphi(t_0, X_1, \ldots, X_l)$ of *QMLO* effectively computable from ψ such that $\varphi(t_0)$ is equivalent (over the reals) to ψ.

2. For every formula $\varphi(t_0, X_1, \ldots, X_l)$ of *QMLO* (with no individual variable except t_0 free) there is a formula $\psi(X_1, \ldots, X_l)$ over the modalities $\Gamma \cup \{\Diamond_1, \overleftarrow{\Diamond}_1\}$ effectively computable from $\varphi(t_0)$ such that $\varphi(t_0)$ is equivalent (over the reals) to ψ.

In particular QTL is expressibly complete for *QMLO* .

Proof:
The first claim is similar to Proposition 2.4. Once the modalities have truth tables in some predicate logic the translation of every formula follows by easy induction.

The second claim is proved by induction on the total number of bounded quantifiers in the *QMLO* formula $\varphi(t_0)$ (we do not display here monadic predicates that might occur in φ). If there are none then $\varphi(t_0)$ is equivalent to some $TL(\Gamma)$ formula since Γ is expressively complete. Otherwise we eliminate some innermost occurrence of a bounded quantifier in φ: Assume $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$ is a subformula of φ with θ in pure monadic logic. Note that t_1 maybe t_0 and that t is the only variable free in θ. Therefore there is some formula $\psi(t_0, X))$ that has one more predicate variable X and one bonded quantifier less than $\varphi(t_0)$ such that φ is obtained from $\psi(t_0, X))$ by substituting $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$ for $X(t_1)$. By assumption there is a $TL(\Gamma)$ formula θ_1 which describes in every model the same predicate as $\theta(t)$ and therefore $\Diamond_1\theta_1$ describes the same predicate as $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$. By induction assumption there is a QTL formula $\psi_1(X)$ which is equivalent to $\psi(t_0, X))$. It is now clear that $\psi_1(\Diamond_1\theta_1)$ describes in every model the same set as $\varphi(t_0)$. □

In particular the expressive power of the quantitative temporal logic does not depend on the choice of the complete set of modalities Γ. Hence in the future we will assume that $\Gamma = \{Until, Since\}$.

4. The Expressive Power of *QMLO*

The two modest modalities $\overleftarrow{\Diamond}_1$ and $\Diamond_1$ when added to the $until$ and the $since$ connectives are quite strong in expressive power. They are capable of expressing modalities concerning intervals other than the open unit interval and they maybe used to express all of the modal operators suggested in the literature for decidable real time temporal logics (see Sect. 4.2). In view of theorem 3.4 we can phrase the expressibility results in terms of the temporal logic QTL or in terms of the predicate logic *QMLO* .

4.1. General Bounded Quantifiers

The modalities $\overleftarrow{\Diamond}_1$ and $\Diamond_1$ refer to the existence of a witness in the next or in the previous (open) unit of time. More general modalities can be defined from these two. In fact any open, closed or half open, bounded or unbounded interval can be the domain of a definable operator.

Notations like $(\exists t)^{<t_0+n+m}_{>t_0+n}$, $(\exists t)^{<\infty}_{>t_0+n}$, $(\exists t)^{\leq t_0+n+m}_{\geq t_0+n}$ are self explanatory. Here are formulas of *QMLO* that define more general bounded quantifiers:

(a) $(\exists t)^{\leq t_0+1}_{\geq t_0} X(t) \equiv X(t_0) \vee (\exists t)^{<t_0+1}_{>t_0} X(t)$

(b) $(\exists t)^{\leq t_0+1}_{>t_0} X(t) \equiv (\exists t)^{<t_0+1}_{>t_0} X(t) \vee [\text{First } (t_0, X) \wedge (\forall t_1)^{<t_0+1}_{>t_0} (\exists t)^{<t_1+1}_{>t_1} X(t)]$

where First (t_0, X) says that there is a first point past t_0 for which $X(t)$ holds:

$$\text{First } (t_0, X) \equiv \exists t_1 [t_0 < t_1 \wedge X(t_1) \wedge \forall t (t_0 < t < t_1 \rightarrow \neg X(t))] \tag{7}$$

(Clearly when $X(t)$ has a first point in the future then the last conjunct in (b) is equivalent to $X(t_0 + 1)$).

(c) $(\exists t)^{<\infty}_{>t_0+1} X(t) \equiv (\forall t)^{\leq t_0+1}_{>t_0} \exists t_1 (t_1 > t \wedge X(t_1))$

(d) $(\forall t)^{<t_0+n+1}_{>t_0+n} X(t) \equiv (\forall t)^{<t_0+n}_{>t_0+n-1} (\exists t_2)^{<t+1}_{>t} (\forall t_1)^{<t_2+1}_{>t_2} X(t_1)$ for $n > 0$

This can be summarized:

Theorem 4.1. The extension L of *QMLO* by the following rules is expressive equivalent to *QMLO* over the canonical model.

if $\varphi(t)$ is an L formula with the only free variable t then the following are L formulae:

1. $(\exists t)^{<t_0+n+m}_{>t_0+n} \varphi(t)$, where n is an integer (possible negative) and m a positive integer.
2. $(\exists t > t_0+n)\varphi(t)$ (denoted also by $(\exists t)^{<\infty}_{>t_0+n} \varphi(t)$) and $(\exists t < t_0+n)\varphi(t)$ (denoted by $(\exists t)^{<t_0+n}_{>-\infty} \varphi(t)$), where n is an integer.
3. Similar to (1) or (2) above with weak inequality replacing one or both occurrences of the strong inequality.

Henceforth, we will freely use these generalized quantifiers which are definable in *QMLO* and also the corresponding modalities like $\Diamond_{[n,m]} X$ which is defined by $(\exists t)^{\leq t_0+m}_{\geq t_0+n} X(t)$, and similarly $\Box_{[n,m]} X$. In this new notation $\Diamond_1 X = \Diamond_{(0,1)} X$ and $\overleftarrow{\Diamond}_1 X = \Diamond_{(-1,0)} X$

4.2. Modalities in Real Time Logics

We define in *QMLO* the main metric modalities that were previously proposed:

1. In [MP93] the main metric modalities used are of the form $\Gamma(X) > n$ where n is a natural number and similar modalities with the inequality reversed and with the inequalities replaced by weak inequalities. The meaning of $\Gamma(X) > n$ is that X was true for the last n units of time. $\Gamma(X) = n$ means that X started exactly n units ago and was true from then till now. In *QMLO* this translates to:

$$[\Gamma(X) > n] \equiv (\forall t)^{<t_0}_{>t_0-n} X(t)$$

$$[\Gamma(X) = n] \equiv (\forall t)^{<t_0}_{>t_0-n} X(t) \wedge ((\forall t)^{<t_0-n+1}_{>t_0-n})(\exists s)^{<t}_{>t-1} \neg X(s)$$

   Thus the modalities of Manna, Pnueli are definable in *QMLO* and QTL.

2. Wilke introduced a relative distance construct "the first time $X(t)$ occurs after t_0 is in distance smaller than, larger than or equal to n" [Wilke94]. Specifically, Wilke uses a parameterized predicate formula $d(X,t) \sim n$ which can be interpreted as a family of truth tables for modalities. These modalities depend on the choice of the distance n and on the choice between equality or one kind of inequality. In *QMLO* these modalities can be defined by:

$$d_{<n}(X) \quad \equiv \quad \mathrm{First}(t_0, X) \wedge (\exists t)^{<t_0+n}_{>t_0} X(t)$$

$$d_{=n}(X) \quad \equiv \quad (\exists t)^{\leq t_0+n}_{>t_0} X(t) \wedge (\forall t)^{<t_0+n}_{>t_0} \neg X(t)$$

3. The popular logic $MITL$ [AFH96] is based on an infinite set of modalities $until_I$ where I is a non-singular interval with integer endpoints - these modalities are called constrained *until* modalities. In one of the different variants of this operator $X\, until l_I\, Y$ means that there is t in the interval I such that in t units of time Y will hold and until then X holds (see also [HRS98] for a different meaning). The constrained *until* modalities are easily expressible in QTL, e.g:

$$X \text{ until}_{(n,n+m)} Y \equiv \Box_{(0,n]}(X \wedge (X \; until \; Y)) \wedge \Diamond_{(n,m)} Y$$

   Similar definitions work for since and for half open or closed intervals. Hence:

   Proposition 4.2. $MITL$ and QTL have the same expressive power.

4. In [HRS98] a modality $\triangleright$ is defined as $\triangleright_{(n,n+m)} X$ - the first instance of $X(t)$ in the future lies in the interval $(t_0 + n, t_0 + n + m)$. This is defined in *QMLO* by:

$$\triangleright_{(n,n+m)} X \equiv \; First(t_0, X) \wedge (\exists t)^{<t_0+n+m}_{>t_0+n} X(t) \wedge \neg(\exists t)^{\leq t_0+n}_{>t_0} X(t)$$

   Note that the conjuncts were defined in section 4.1

5. Timer Normal Form

Our analysis and decidability results are based on the elimination of the metric; reducing the metric case to the non metric one and using theorem 2.2. To this end we shall define the class of formulas in **timer normal form**. First order and second order formulas in timer normal form are formulas in first order and second order predicate calculus in which the metric occurs only in a very restricted form. We shall later see that as far as satisfiability is concerned every formula in timer normal form can be replaced by a non metric formula. On the other hand many metric formulas (and in particular all the formulas of *QMLO*) are equivalent to formula of timer normal form.

Definition 5.1. (Timer formulas)
We call the following *QMLO* formula *a simple Timer Formula*

$$\mathrm{Timer}(X,Y) \equiv \forall t(Y(t) \longleftrightarrow (\forall t_1)^{<t}_{>t-1} X(t_1))$$

we say that Y is *a timer for* X. For every n we define the formula:

$$\mathrm{Timer}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \equiv \bigwedge_i Timer(X_i,Y_i)$$

In the formula $\mathrm{Timer}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n)$ each Y_i is a timer that measures if X_i persisted for the duration of the last unit of time. Note that $\mathrm{Timer}(X,Y)$ is equivalent to the QTL formula:

$$Always((Y \longleftrightarrow \overleftarrow{\Box}_1 X)).$$

(The modality $Always$ was defined in Example 2.3.)

Definition 5.2. (Timer Normal Forms)
A formula is said to be in first order, second order or temporal **timer normal form** if it has the form

$$\exists \overline{W}(\mathrm{Timer}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi),$$

where $\overline{W}$ is a list of monadic variables and φ is first order monadic, second order monadic or temporal formula, respectively.

Remark 5.3. (Reducing the future to the past)
The next theorem will show that every *QMLO* formula is equivalent to a formula in timer normal form. To this end we shall show first that $\diamondsuit_1$ is expressible by a formula in timer normal form (note that the modality $\diamondsuit_1$ is not expressible by a formula in $TL(until, since, \overleftarrow{\diamondsuit}_1)$; the existential quantification over sets is essential in expressing $\diamondsuit_1$ in timer normal form). To do this we observe that if

$$Always(Y \leftrightarrow \diamondsuit_1 X)$$

then

1. If Y holds at t then Y holds until the next occurrence of X.
2. If X holds at t then Y was true for the previous unit of time and

3. If X was false for a unit interval of time then Y was false in the beginning of the interval (hence somewhere in the interval).

Hence, $Always(Y \leftrightarrow \Diamond_1 X)$ implies the following formula ψ

$$\psi \equiv Always(Y \rightarrow Y\,until\,X) \wedge Always(X \rightarrow \overleftarrow{\Box}_1 Y) \wedge Always((\overleftarrow{\Box}_1 \neg X) \rightarrow \overleftarrow{\Diamond}_{\leq 1} \neg Y).$$

Moreover, the first and the third conjuncts in ψ imply that $Y \rightarrow \Diamond_1 X$; the second conjunct implies $\Diamond_1 X \rightarrow Y$.

Therefore, ψ is equivalent to $Always(Y \leftrightarrow \Diamond_1 X)$. Hence,

$$\Diamond_1 X \text{ is equivalent to } \exists Y(Y \wedge \psi).$$

It remains to show that ψ is equivalent to a formula in timer normal form.

Note that all metrical modalities in ψ are past modalities. We construct the timer normal form for ψ in two steps.

First we get rid of the weak order in $\overleftarrow{\Diamond}_{\leq 1} \neg Y$. Similar to the equation (b) of Sect. 4.1 we have that $\overleftarrow{\Diamond}_{\leq 1} \neg Y$ is equivalent to $\overleftarrow{\Diamond}_1 \neg Y \vee (last(\neg Y) \wedge \overleftarrow{\Box}_1 \overleftarrow{\Diamond}_1 \neg Y)$, where the modality $last(Z)$ holds at t iff there is a last occurrence of Z before t (it can be expressed as $\neg Z since Z$). Hence, ψ is equivalent to

$$Always(Y \rightarrow Y\,until\,X) \wedge Always(X \rightarrow \overleftarrow{\Box}_1 Y) \wedge Always((\overleftarrow{\Box}_1 \neg X) \rightarrow (\overleftarrow{\Diamond}_1 \neg Y \vee (last(\neg Y) \wedge \overleftarrow{\Box}_1 \overleftarrow{\Diamond}_1 \neg Y))).$$

Using the law $\overleftarrow{\Diamond}_1 Z \leftrightarrow \neg \overleftarrow{\Box}_1 \neg Z$ we can rewrite the last formula as

$$Always(Y \rightarrow Y\,until\,X) \wedge Always(X \rightarrow \overleftarrow{\Box}_1 Y) \wedge Always((\overleftarrow{\Box}_1 \neg X) \rightarrow (\neg \overleftarrow{\Box}_1 Y \vee (last(\neg Y) \wedge \overleftarrow{\Box}_1 \neg \overleftarrow{\Box}_1 Y))).$$

Finally, introducing a timer for every subformula that starts with $\overleftarrow{\Box}_1$ we can easily see that ψ is equivalent to the following formula in timer normal form:

$$\exists T_1 T_2 T_3 . Timer(\neg X, T_1) \wedge Timer(Y, T_2) \wedge Timer(\neg T_2, T_3) \wedge$$
$$\wedge Always(Y \rightarrow Y\,until\,X) \wedge Always(X \rightarrow T_2) \wedge Always(T_1 \rightarrow (\neg T_2 \vee (last(\neg Y) \wedge T_3)).$$

(Here $Timer(\neg Z, W)$ stands for $\exists Z'.Timer(Z', W) \wedge Always(Z' \leftrightarrow \neg Z)$.)

Theorem 5.4. (Reduction to Timer Normal Form)
There is an algorithm which associates with any formula $\varphi(\bar{t}, \overline{Z})$ of *QMLO* variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and a formula $\overline{\varphi}(\bar{t}, X_1, \ldots, X_n, Y_1, \ldots, Y_n, \overline{Z})$ of pure first-order MLO such that $\varphi(\bar{t}, \overline{Z})$ is satisfiable iff the following formula is satisfiable:

$$\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \overline{\varphi}(\bar{t}, \overline{X}, \overline{Y}, \overline{Z})$$

Moreover, $\varphi(\bar{t}, \overline{Z})$ is equivalent to the formula:

$$\exists \overline{X} \exists \overline{Y} (\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \overline{\varphi}(\bar{t}, \overline{X}, \overline{Y}, \overline{Z}))$$

Proof:
(Sketch) The proof is by induction on the number of metric quantifiers in φ. If there are none then φ is already in pure MLO form. Else we may focus on an innermost metric quantifier and we assume first that it is of the form $(\forall t)^{<t_1}_{>t_1-1}\theta(t)$ so that there is some formula $\psi(\bar{t}, \overline{Z}, Y))$ that has one more predicate variable Y and one bonded quantifier less than φ such that φ is obtained from $\psi(\bar{t}, \overline{Z}, Y)$ by substituting $(\forall t)^{<t_1}_{>t_1-1}\theta(t)$ for $Y(t_1)$. It is clear that φ is equivalent to

$$\exists Y\, X(\psi \wedge \mathrm{Timer}(X, Y) \wedge \forall t.X(t) \leftrightarrow \theta(t)),$$

where X is a fresh variable. By the inductive assumption ψ has a timer normal form $\bar{\psi}$. I.e, $\psi(\bar{t}, \overline{Z}, Y)$ is equivalent to the formula:

$$\exists \overline{X} \exists \overline{Y} (\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \overline{\psi}(\bar{t}, \overline{X}, \overline{Y}, \overline{Z}, Y))$$

Hence φ is equivalent to the following formula in timer normal form

$$\exists \overline{X} \exists \overline{Y} \exists Y \exists X (\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \mathrm{Timer}(X, Y) \wedge \overline{\psi} \wedge \forall t.X(t) \leftrightarrow \theta(t)).$$

If the innermost simple metric formula that we chose starts with a "future quantifier" $(\forall t)^{<t_1+1}_{>t_1}$ it is dealt similarly using Remark 5.3. □

6. Elimination of the metric

Next we show how to eliminate the timer formulas from formulas in timer normal form. This will reduce the questions of decidability and complexity to the similar questions for the non metric case. With each timer formula we associate a pure MLO formula that reflects much of the nature of the Timer. Specifically, with the formula $\mathrm{Timer}_n(X_1, \ldots X_n, Y_1, \ldots, Y_n)$ we associate a pure monadic first order formula $\overline{\mathrm{Timer}}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$. This formula is a conjunction of a few simple formulas that reflect the following properties:

1. Every predicate Y_i is a "timer". Specifically, for every i the formula A_i will say that
   (a) Y_i is true at zero and
   (b) Y_i has finite variability and
   (c) every maximal interval I, where Y_i is true has either the form $[a, b]$ or $[a, \infty)$ İn other words, the set of points where Y_i holds is a closed set.
2. The "timer" Y_i times the predicate X_i: For every i the formula B_i says that
   (a) if Y_i holds at a point other then zero then X_i has been true for a while just before that point.
   (b) if X_i is true from a point c forever then Y_i will be true from some (possibly later) point forever.
   (c) if Y_i is true at b and X_i is true in the interval $[b, c)$ then Y_i is true at c.
3. The "timers" Y_i and Y_j for two different predicates are consistent with each other: The formula $C_{i,j}$ says that

(a) if at b Y_i is true and Y_j is false, then there is an interval $I = (c, b)$ with the following properties: X_i is true on I and there is a point in the interval where X_j is false.

(b) if both Y_i and Y_j became true at a point b then for every previous point c: if X_i is true on the interval (c, b) then X_j is true there.

It is clear that all the properties above can be written as formulas in first-order monadic logic of order (finite variability is the only one that is not immediate and it was formulated in section 2.1). Note that $X_1, \ldots, X_n$ are arbitrary and not just finite variability predicates.

Definition 6.1. $\overline{\mathrm{Timer}}_n$ is the pure first-order MLO formula which is the conjunction of the formulas $A_i, B_i, C_{i,j}$ above:

$$\overline{\mathrm{Timer}}_n \equiv \bigwedge_i (A_i \wedge B_i) \wedge \bigwedge_{i,j} C_{i,j}$$

The following theorem establishes the connection between the formula Timer of *QMLO* and the formula $\overline{\mathrm{Timer}}$ of pure MLO.

Theorem 6.2. (Elimination of the metric)
The predicates $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ over R^+ satisfy the formula $\overline{\mathrm{Timer}}_n$ if and only if there is an order preserving bijection $\rho : R^+ \to R^+$ such that $P_1\rho, \ldots, P_n\rho,\ Q_1\rho, \ldots, Q_n\rho$ satisfy Timer_n ($P_i\rho$ is the image of P_i under the bijection ρ, i.e $(P\rho)(\tau)$ iff $P(\rho(\tau))$).

Proof:
The detailed proof is very technical and can be found in [HR03]. □

Order preserving bijections preserve pure (non metric) formulas; for every first or second order monadic formula $\varphi(X)$, P satisfies $\varphi(X)$ iff $P\rho$ satisfies $\varphi(X)$. Hence:

Theorem 6.3. For every first-order or second-order monadic formula φ:

$$\overline{\mathrm{Timer}}_n \wedge \varphi \text{ is satisfiable iff } \mathrm{Timer}_n \wedge \varphi \text{ is satisfiable.}$$

7. Decidability

With the reduction of metric logics to timer normal form (theorem 5.4) and with the reduction of timer normal form formulas to non metric formulas (theorem 6.2) we obtain a reduction of metric formulas to non metric formulas. Thus the following theorem and corollary follow from the similar theorems for non metric logic (theorem 2.2).

Theorem 7.1. (Decidability for timer normal form)

1. The satisfiability of the formulas in first-order timer normal form in the canonical model is decidable.

2. The satisfiability of the formulas in the second-order timer normal form under the finite variability interpretation is decidable.

Corollary 7.2. 1. Satisfiability in the canonical model is decidable for *QMLO* and for QTL.

2. Satisfiability in the finite variability canonical model is decidable for *QMLO* and for QTL.

Note that the theorem proves for the first time decidability for the model with unrestricted variability and gives a simple automata free proof for the finite variability model.

8. The Complexity of QTL

In this section we show how one can derive easily the complexity bounds for temporal logic without appealing to automata theoretical techniques.

We start with the observation that the formula $\overline{Timer}$ of the predicate logic (see section 6) is expressible in Temporal Logic, by a simple formula:

Lemma 8.1. There is a TL formula $timer_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ of size $O(n^2)$ over the modalities Until and Since such that $\overline{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is equivalent over the canonical model to $timer_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$.

Proof:
Just observe that every clause $A_i,\ B_i$ and $C_{i,j}$ in the definition of $\overline{Timer_n}$ (see section 6) are expressible by a temporal logic formula which is independent of n. □

Next we want to show how a QTL formula translates into a TL formula that is satisfiable if and only if the original formula is. There is an intermediate step similar to that played by timer normal form for predicate logic. The role of the timer formula is taken by the QTL formulas $Always(X \leftrightarrow \Diamond_1 Z)$ and $Always(X \leftrightarrow \overleftarrow{\Diamond} Z)$. Recall that $Always(X)$ is the modality that states that X always holds with the truth table: $t_0 = t_0 \wedge \forall t.\ X(t)$.

Lemma 8.2. For every QTL formula φ with k metrical modalities there is a TL formula ψ and a subset F of $\{1, \ldots, k\}$ such that the size of ψ is linear in the size of φ and φ is satisfiable if and only if the conjunction of ψ with $\bigwedge_{i \in F} Always(X_i \leftrightarrow \Diamond_1 Z_i) \wedge \bigwedge_{i \in \{1,\ldots,k\} \setminus F} Always(X_i \leftrightarrow \overleftarrow{\Diamond} Z_i)$ is satisfiable.

Proof:
(Sketch) The proof is by induction on k. If there are no metric quantifiers we are done. Assume the claim holds for less than k quantifiers and that φ has k quantifiers. Let (say) $\Diamond_1 \psi$ be an innermost occurrence of a metric modality so that ψ is non metric (the proof is similar when the metric modality is different). Thus there is a formula $\theta(X)$ with $k-1$ metric modalities such that φ is obtained from it replacing X by $\Diamond_1 \psi$. Then φ is satisfiable iff the following formula is satisfiable:

$$\theta(X) \wedge Always(Z \leftrightarrow \psi) \wedge Always(X \leftrightarrow \Diamond_1 Z).$$

The formula $Always(Z \leftrightarrow \psi)$ is non metric, and replacing θ according to the induction assumption and rearranging the formula we obtain the required normal form formula. □

Next we show that the metric formulas that remain in the last lemma are expressible using the Timer formulas:

Lemma 8.3. 1. $Always(X_i \leftrightarrow \Diamond_1 Z_i)$ is equivalent to a formula in timer normal form.

2. The formula $Always(X_i \leftrightarrow \overleftarrow{\Diamond}_1 Z_i)$ is equivalent to $Timer_1(\neg X_i, \neg Z_i)$.

Proof:
The second claim is obvious, while the first one needs some work similar to Remark 5.3 . □

The above lemmas and Theorem 6.2 imply

Theorem 8.4. There is a polynomial time algorithm that for every QTL formula φ of size m with k metrical modalities constructs a TL formula ψ of size $O(k^2 \times m)$ such that (1) ψ is satisfiable in the canonical model iff φ is satisfiable in the canonical model. (2) ψ is satisfiable in the finite variability canonical model iff φ is satisfiable in the finite variability canonical model.

This completes the reduction of QTL to TL. As for the complexity of TL over the standard real model we have again a more general theorem:

Theorem 8.5. [Rab98a] Let Γ be any finite set of modalities definable in monadic first-order logic of order. The satisfiability problem for TL formulas over Γ in the *finite variability* canonical model is in PSPACE.

Theorem 8.6. [Re99] Let Γ be any finite set of modalities definable in monadic first-order logic of order. The satisfiability problem for TL formulas over Γ in the *canonical model* is in PSPACE.

We note here that the proof of the above theorems is automata free and relies on the compositional method [She75].

Corollary 8.7. 1. The satisfiability problem for QTL in the finite variability canonical model is in PSPACE.

2. The satisfiability problem for QTL the canonical model is in PSPACE.

Let φ be an $MITL$ formula (see Sect. 4.2) with m boolean and temporal connectives and let c be the largest constant that appears as a subscript of the bounded *Until* or *Since* operator in φ. From the laws of Section 4 and an unnesting procedure, one can construct a QTL formula ψ of size $O(m \times c)$ such that ψ is satisfiable (in either the canonical or the finite variability canonical model) iff φ is satisfiable in that model. Hence,

Corollary 8.8. 1. The satisfiability problem for $MITL$ in the finite variability canonical model is in PSPACE.

2. The satisfiability problem for $MITL$ in the canonical model is in PSPACE.

Corollary 8.8(1) reproves (and improves) the result of [AFH96].

9. Strong Logics which are Normalizable

Using much heavier machinery we are able to reduce stronger logics to timer normal form and deduce their decidability. In terms of metric predicate logic there is a natural logic which has strong expressive power and is still decidable. We denote it by *Q2MLO*. It looks deceptively similar to *QMLO* but later sections will show that it is significantly stronger than *QMLO* . There is no natural quantified temporal logic that extends QTL and seems strong enough to include everything that we could wish for. Instead we describe a mechanism to define temporal logics which are strong yet decidable. This mechanism uses truth tables for the temporal connectives in the proper predicate logic; *Q2MLO* or even some restricted second order logic. It is worthwhile to notice that unlike *QMLO* and QTL, here we shall encounter temporal logics which are decidable for the class of finite variability models and undecidable for the class of general models.

In section 2.3 we defined what does it mean that an operator is defined by a truth table. A more general notion is that of an operator being definable by a generalized truth table.

Definition 9.1. (Generalized Truth Tables)
A formula $\overline{O}(Y, X_1, \ldots, X_k)$ in the predicate logic L is a *generalized truth table* for the modality $O^{(k)}$ if for every structure M and subsets $P_1, , \ldots, , P_k,\ Q$ of the universe $|M|$ of M

$$O_M(P_1, \ldots, P_k) = Q \text{ if and only if } M \models_{MLO} \overline{O}[Q, P_1, , \ldots, , P_k].$$

Remark 9.2.

1. If $\psi(Y, X_1, \ldots, X_k)$ is a generalized truth table then for every structure M and subsets $P_1, \ldots, P_k$ of the universe $|M|$ of M there is a unique $Q \subseteq |M|$ such that $M \models \psi[Q, P_1, \ldots, P_k]$.
2. If $\psi(t_0, X_1, \ldots, X_n)$ is a truth table for a modality O then the formula $\forall t_0(Y(t_0) \longleftrightarrow \psi(t_0, X_1, \ldots, X_n)$ is a generalized truth table for O.

The following proposition shows how to define temporal logics which are reducible to timer normal form formulas:

Proposition 9.3. If every modality in the temporal logic TL has a generalized truth table in the first-order (respectively, second-order) timer normal form then for every formula $\varphi(X_1, \ldots, X_n)$ of TL there corresponds a formula $\overline{\varphi}(Y, X_1, \ldots, X_n)$ in first-order (respectively, second-order) timer normal form, which is effectively constructed from the generalized truth tables for the modalities that occur in φ such that for every M, and predicates $Q, P_1, \ldots, P_n$

$$Q = \{\tau \ : \ \langle M, \tau, P_1, \ldots, P_n\rangle \models_{TL} \varphi\} \quad \text{iff} \quad M \models_{MLO} \overline{\varphi}[Q, P_1, \ldots, P_k].$$

As a consequence of Proposition 9.3 and Theorem 7.1 we obtain

Corollary 9.4.

1. If every modality in the temporal logic TL has a generalized truth table in the first-order timer normal form then the satisfiability problem for TL is decidable.
2. If every modality in the temporal logic TL has a generalized truth table in the second-order timer normal form under finite variability interpretation then the satisfiability problem for TL is decidable.

The following technical theorem is the key for finding interesting modalities with a generalized truth table in timer normal form. By Corollary 9.4 the temporal logics with such modalities are decidable. The theorem states that certain temporal operators with *truth table* in a metric logic that is wider than *QMLO* are still definable by a *generalized truth tables* in timer normal form. Note that the theorem involves two different notions of definability: definability by a truth table (considered in Section 2.3) and definability by a generalized truth table.

Theorem 9.5. (generalized truth tables)

1. Let $\varphi(t_0,\ t, X_1, \ldots, X_l)$ be a formula of first-order MLO and $m < n$ be integers. The operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t, X_1, \ldots, X_l)$ is definable by a generalized truth table in first-order timer normal form, which is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.

2. Let $\varphi(t_0,\ t, X_1, \ldots, X_l)$ be a formula of second-order MLO and $m < n$ be integers. The operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t, X_1, \ldots, X_l)$ is definable by a generalized truth table ψ in second-order timer normal form, moreover, ψ is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.

3. Let $\varphi(t_0,\ t, X_1, \ldots, X_l)$ be a formula of second-order MLO and $m < n$ be integers. For the semantics of finite variability predicates the operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t, X_1, \ldots, X_l)$ is definable by a generalized truth table ψ in second-order timer normal form, moreover, ψ is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.

About the proof: Given an interval I and monadic predicates on I we denote by $Th^n(I)$ the set of all sentences with quantifier depth n that are true in I with these predicates. The key observation for (1) is that Th^n of a lexicographical sum[1] of two intervals is definable from the Th^n theory of the summands [Ehr61]. For (2) and (3) we use a similar theorem for monadic second order logic [She75].

In view of the last theorem it is natural to identify a fragment of first order logic which is larger than *QMLO* and still lends itself to reduction to timer normal form:

Definition 9.6. (The fragment *Q2MLO*)
Q2MLO is the fragment of MLO^+ which is built from the atomic formulas $t_1 < t_2, t_1 = t_2, X(t)$ using Boolean connectives, first order quantifiers and the following rule:

if $\varphi(t_0, t)$ is a formula of *Q2MLO with* $t_0,\ t$ *its only first order free variables*,
m and n integers, and $m < n$ then the formulas
$(\exists t)^{<t_0+n}_{>t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{>t_0+m}\varphi$ are in *Q2MLO* .

Note that the only differences between *Q2MLO* and *QMLO* are

1. *QMLO* uses only metrical quantifiers $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$, whereas in *Q2MLO* we allow more general bounds on the metrical quantifiers

[1] Let M_1 and M_2 be two linear orders with disjoint universes. Their lexicographical sum is the structure over the union of their universes; the elements of M_1 precede the elements of M_2

2. after a metrical quantifier in *QMLO* only a formula with one free variable may appear, whereas in *Q2MLO* it may contain two variables one of which must appear in the bounds of the quantifier.

The similarity between the definition of *QMLO* and of *Q2MLO* should not mislead the reader; the extra free variable in the quantified formula makes the expressive power of *Q2MLO* significantly bigger than that of *QMLO* . The more liberal bounds on the quantifiers are needed since it is not clear if they can be expressed using only the modest bounds 1 and -1 as in *QMLO* (theorem 4.1). It is worth mentioning that there is no point in trying to squeeze the method further and allow another free variable as the logic that allows a third free variable in the bounded formula is already undecidable.

Theorem 9.7. Satisfiability in the canonical model is decidable for *Q2MLO*. Satisfiability in the finite variability canonical model is decidable for *Q2MLO*.

Theorem 9.8. (decidable temporal logics)

1. Assume that every connective of a temporal logic T has a truth table of the form $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t,\ X_1,\ldots,X_l)$, where $\varphi(t_0,\ t,\ X_1,\ldots,X_l)$ is a formula of first-order MLO and $m < n$ are integers. Then T is decidable. Similarly for the tables with one or two weak inequalities replacing the strict ones.

2. Assume that every connective of a temporal logic T has a truth table definable in *Q2MLO* . Then T is decidable.

3. Assume that every connective of a temporal logic T has a truth table of the form $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t, X_1,\ldots,X_l)$, where $\varphi(t_0,\ t, X_1,\ldots,X_l)$ is a formula of second-order MLO and $m < n$ are integers. Then T is decidable under finite variability semantics. Similarly for the tables with one or two weak inequalities replacing the strict ones.

10. Decidable Extension of QTL

In this section we use the logic *Q2MLO* and the previous theorems to produce a host of decidable temporal logics that extend QTL. These examples include to the best of our knowledge every extension that was suggested in the literature as well as new logics that seem natural in our setting.

10.1. A sequence of decidable extensions of QTL

Our first example involves a formula that was often discussed as a probable example of a formula that cannot be expressed in the minimal metric logics like QTL or $MITL$. We denote this formula by $(\diamond X_1 \diamond X_2)_1$. It says that within the next unit of time first X_1 and then X_2 will occur. In the monadic logic of order with the successor function, and indeed in *Q2MLO* this is defined by the formula $(\exists t)^{<t_0+1}_{>t_0}.\exists t_1 t_2(t_0 < t_1 < t_2 < t \wedge X_1(t_1) \wedge X_2(t_2))$. This example serves beautifully the claim that the proper way to develop the theory is within the framework of monadic logic and not via additions to temporal logics. We note first that there is a simple generalization $(\diamond X_1 \ldots \diamond X_n)_1$ which is defined by the *Q2MLO* formula $(\exists t)^{<t_0+1}_{>t_0}.\exists t_1 t_2 \cdots t_n(t_0 < t_1 < t_2 < \cdots < t_n < t \wedge X_1(t_1) \wedge X_2(t_2) \wedge \cdots \wedge X_n(t_n))$. Next we denote for the moment by $QTL(n)$ the temporal logic QTL extended by the first n modalities just defined (in fact every modality in the list can be expressed using the next one and substituting $True$

for one of the variables. Therefore $QTL(n)$ can be obtained by adding one modality to QTL). We will show that this is a strict hierarchy; the new modality in $QTL(n)$ is not definable in $QTL(n-1)$. In particular the fact that $QTL(n+1)$ is decidable does not follow from the fact that $QTL(n)$ is decidable, and certainly not from the fact that QTL is decidable, and it has to be proved. In contrast the mere fact that these modalities have a truthtable in *Q2MLO* proves that these logics are decidable. Here is the outline of the formal development:

The following proposition shows that this is a *proper* extension of QTL

Proposition 10.1. The connective $(\diamond X_1 \diamond X_2)_1$ is not expressible in QTL (or equivalently in $MITL$).

Proof:
(Sketch) Let $X(t)$ be a predicate that is true exactly at the points $n \times \frac{2}{3}$ for all natural numbers n. It can be shown by structural induction that every QTL formula concerning X is equivalent from some point t_0 on to either X or to $\neg X$ or to $true$ or to $false$. On the other hand the formula $(\diamond X \diamond X)_1$ does not have this property. □

The construction above generalizes to yield a proper increasing hierarchy of connectives:

Theorem 10.2. (Hierarchy of temporal logics)
For every n let $(\diamond X_1 \ldots \diamond X_n)_1$ be the n-place modality that states that within one unit of time in the future there are points $t_1 < \cdots < t_n$, such that $X_i(t_i)$ for all $i \leq n$. Let $QTL(n)$ be the temporal logic $TL(Until, Since, (\diamond X_1 \ldots \diamond X_n)_1)$. Then:

1. For every n the logic $QTL(n)$ is decidable.

2. These logics form a strict hierarchy; for every n the logic $QTL(n+1)$ is strictly more expressible than $QTL(n)$.

10.2. A one place modality hierarchy

It is worth noting that a hierarchy similar to the above can be obtained without increasing the arity of the new modalities. Let $M_n(X)$ be the modality that says that within the next unit of time there will be at least n points satisfying X. Similarly to the theorem 10.2 we have:

the temporal logic QTL with the modalities $M_1(X), \cdots, M_n(X)$ is incapable of expressing $M_{n+1}(X)$.
Therefore we get again a strict hierarchy:

Theorem 10.3. (Hierarchy of unary temporal logics)
For every n let $QTL'(n)$ be the temporal logic obtained extending QTL by the modalities $M_1(X), \cdots, M_n(X)$. Then

1. For every $n > 0$ the logic $QTL'(n)$ is decidable.

2. These logics form a strict hierarchy; for every $n > 0$ the logic $QTL'(n+1)$ is strictly more expressible than $QTL'(n)$.

10.3. Automata Modality

We provide some definitions about finite variability signals and thereafter define a variety of connectives which satisfy the premises of theorem 9.5 and therefore yield decidable temporal logics.

A string $l_0 l_1 \dots l_n$ is **stuttering free** iff it is non empty and no two adjacent symbols in it are equal.

Recall that a function η from a bounded subinterval $(a,\ b)$ of the reals into a finite set Σ has *finite variability* if there exists a finite increasing sequence $a = a_0 < a_1 \cdots < a_n = b$ such that η is constant on every interval (a_i, a_{i+1}).

A function f from R^+ to Σ has finite variability if for every bounded interval I the restriction of f to I has finite variability.

The following lemma is straightforward.

Lemma 10.4. Suppose that $\eta :\ (a,\ b) \to \Sigma$ has finite variability, then there exists a unique increasing sequence $a = a_0 < a_1 < a_2 \cdots < a_n = b$ such that

1. η is almost constant on every interval $(a_i,\ a_{i+1})$, i.e., for every i there is $l_i \in \Sigma$ such that the set $\{x \in (a_i,\ a_{i+1}) \ :\ \eta(x) \neq l_i\}$ is finite.

2. For every $i < n - 1$, the value of η on (a_i, a_{i+1}) differs from the value of η on (a_{i+1}, a_{i+2}).

Definition 10.5. (Trace of a finite variability function.) Let η be a finite variability function over $(a,\ b)$ and let $a_0, \dots, a_n$ be as in Lemma 10.4. For $i < n$ let l_i be the values of η on (a_i, a_{i+1}). The trace of η (denoted by **trace**(η)) is the stuttering free string $l_0 l_1 \dots l_{n-1}$.

Trace of a tuple: Let $\langle \eta_1, \dots, \eta_n \rangle$ be an n-tuple of finite variability functions from $(a,\ b)$ into $\{0,\ 1\}$. With this n-tuple we associate a function η from $(a,\ b)$ into $\{0, 1, \dots, 2^n - 1\}$ defined as $\eta(t) = i$ if $\langle \eta_1(t), \dots, \eta_n(t) \rangle$ is the binary representation of i. The above mapping defines a one-to-one correspondence between the set of n-tuple of finite variability functions from $(a,\ b)$ into $\{0, 1\}$ and finite variability functions from $(a,\ b)$ into $\{0, \dots, 2^n - 1\}$. The trace of an n-tuple (denoted by **trace**$(\eta_1, \dots, \eta_n)$) is defined as the trace of the corresponding function.

Let R be a regular expression over the alphabet $\{0,\ 1\}^n$ and let $k < m$ be integers. Define the n-ary connective $R_{k,m}$ as follows: $R_{k,m}(X_1, \dots, X_n)$ holds at t_0 if there is $t' \in (t_0 + k,\ t_0 + m)$ such that **trace**$(Y_1, \dots, Y_n) \in R$, where Y_i is the restriction of X_i to $(t_0,\ t')$.

Proposition 10.6. The extension of QTL by the connectives $R_{k,m}$ is decidable for finite variability semantics.

Proof:
(Sketch) It is straightforward to show that for every regular expression R there is a formula $\varphi_R(t_0,\ t, \bar{X})$ in pure monadic second order logic such that $\varphi_R(t_0,\ t, \bar{X})$ holds iff **trace**$(Y_1, \dots, Y_n) \in R$, where Y_i is the restriction of X_i to $(t_0,\ t)$. Hence, by Theorem 9.8, the logic is decidable. □

10.4. Decidable Modalities about a Whole Interval

The next collection of examples shows that decidable metric modalities can be defined which describe the behavior on the whole interval $(t_0,\ t_0+1)$ (as opposed to claiming the existence of a nice subinterval).

Let s be a stuttering free string over the alphabet $\{0,\ 1\}^n$ and let $k < m$ be integers. Define an n-ary connective s_m as follows: $s_m(X_1, \ldots, X_n)$ holds at t_0 if **trace**$(Y_1, \ldots, Y_n) = s$, where Y_i is the restriction of X_i to $(t_0,\ t_0 + m)$.

Proposition 10.7. The extension of QTL by the connectives s_m is decidable.

Proof:
(Sketch) We will show that the connective s_m is definable in *Q2MLO* and therefore, the extension is decidable. It is easy to write a first order monadic formula $\varphi_1(\bar{X}, t_0, t)$ which says that the trace of $\overline{X}$ on the interval $(t_0,\ t)$ is s; it is also easy to write a first order monadic formula $\varphi_2(\bar{X}, t_0, t)$ which says that the trace of $\overline{X}$ on the interval $(t_0,\ t)$ has length greater than the length of s. The desirable formula is:

$$(\exists t)^{<t_0+n}_{>t_0}\varphi_1 \wedge \neg(\exists t)^{<t_0+n}_{>t_0}\varphi_2.$$

□

11. Comparison with previous work

11.1. Previous approach

We try to give a short factual account of the previous approach to metric logics and its technical achievements. Later we shall also indulge in a more critical discussion of the models and formalisms that were chosen.

Before the metric was taken into account the standard real line served as the predominant model for systems evolving in continuous time [GHR94] and [BKP85]. Metric was introduced into the language in different forms around 1990 or shortly before as described in [AH92]. For a reason obscure to us this was accompanied by abandoning the canonical model and introducing instead models which are sequences of points in time or of intervals of time. Specifically these models were:

Definition 11.1. (The State Sequence Model)
A state sequence is an ω-sequence (t_0, σ_0), ..., (t_n, σ_n), ... where t_i is an unbounded increasing sequence of reals and σ_i gives the values of the monadic predicates at t_i. Monotonic state sequences are those where the sequence t_i is strictly monotone and weakly monotonic state sequences require only weak monotonicity.

Definition 11.2. (The Interval Sequence Model)
An interval sequence is an ω-sequence $(I_0, \sigma_0), \ldots, (I_n, \sigma_n) \ldots$ where I_i are disjoint intervals that cover R^+ and I_i precedes I_{i+1} and σ_i gives the values of the monadic predicates on I_i.

Every interval sequence describes in a natural way a flaw with finite variability over the canonical real model and in [AFH96] it is implied that this is the intended model (remark 2.3.2.2 in [AFH96]). For state sequences there is no obvious canonical model in the background but there are different classes of real time flaws that can be chosen to correspond to the monotone state sequences.

The logic used for reasoning about the models before considering the metric was mainly classical predicate logic or temporal logic as defined within the framework of predicate logic [Kamp68, GPSS80, GHR94]. In contrast the introduction of the metric was accompanied by almost a total abandoning of the

predicate logic. The means to deal with the metric were added directly to temporal logic. This resulted in a large number of formalisms, notably among them some different dialects of $MITL$ [AH92, AFH96]. For these formalisms it was shown that:

1. the formalisms dealing only with future were basically equivalent to each other.
2. The formalisms are decidable with PSPACE complexity. The proof was by reduction to the non emptiness problem for timed automata [AD94].
3. Extending the formalism in a way that enables the statement "P will hold in one unit of time" results in an undecidable formalism. In contrast "P will *first* hold in one unit of time" is expressible in the formalism [AFH96].
4. Statements about intervals with arbitrary integer endpoints can be phrased in a formalism that addresses only unit intervals [RSH98].

These facts can be stated and proved with some care for finite variability flaws over the real line. Therefore these results for our QTL are new only in that they apply also to arbitrary (non finite variability) models.

A novel approach was taken by Wilke [Wilke94]. He introduced a strong (second order) predicate logic that was enough to express all the temporal logics above and more. This logic is decidable and corresponds to timed automata in the way that pure second order logic corresponds to finite automata. Thus Wilke preceded us in suggesting a strong decidable logic as the realm for specifying decidable temporal predicates. Wilke's work addresses state sequences but it can be modified to apply to finite variability canonical flaws. Thus again the technical novelty in our work is its applicability to non finite variability flows.

Our aim was mainly to get the theory of real time logic back to mainstream mathematical logic: to use the real line itself as the canonical model and to use classical predicate logic as a framework. Within this framework finite variability flaws and general flaws are treated uniformly, and a natural simple metric temporal logic QTL emerges and is shown to absorb the previously defined metric constructs. We proved that this logic is decidable for finite variability flaws as well as for general flaws by a reduction to the non metric case. This reduction was then extended to prove decidability of a larger predicate logic and we suggest this logic as hunting grounds for new natural decidable modalities.

11.2. Decidability and Complexity Techniques

Alur and Dill introduced timed automata and proved that the emptiness problem is decidable for the class of these automata [AD94]. The proof reduces the problem to the emptiness problem for Büchi ω-automata. This result was used in all the previous approaches to prove that the logic is decidable (we suspect that this is the main reason why all the authors ignored the standard model in favor of an ω sequence model of some kind).

The class of languages accepted by timed automata is not closed under complementation. On the other hand logic is closed under negation. Therefore the usual way to show that a logic was decidable was to introduce some variations and restriction of timed automata which is closed under complementation and providing an effective transformation from logical formulae to "equivalent" automata in this class. These classes were given nice descriptive names like "Recursive Event Clock Automata" [HRS98] or

"Bounded Two Way Deterministic Timed Automata" [AH92a]. It seems that these classes of automata do not give any additional insight about the corresponding logic.

In contrast our decidability proof remains within the framework of logic, first reducing the language to a normal form ("Timer normal form") and then reducing the problem to the known result about the decidability of pure MLO (or TL). In particular our proof applies just as well to the standard model with all the unary predicates (and not just finite variability predicates).

It may have been hoped that the use of automata theory for decidability would provide a less complex procedure and clearer complexity bounds than decidability proofs based on general logical reductions. We found this not to be the case; our method easily yields the same upper bounds or better ones (see section 8).

11.3. Previous Approach - Critical Remarks

In the last fifteen years a vast amount of research was invested in attempts to introduce real time metric formalisms (see [AH92] and [Wilke94]).

> "Recent research in reactive and hybrid systems is dominated by a plethora of Concepts, Terminology, Notations. Unfortunately, this background is not free of ad-hoc and ambiguous decision which are liable to misjudgments and to infliction of myths into the area" [Trak99].

> "The number of formalisms that purportedly facilitate the modelling, specifying and proving of timing properties for reactive systems has exploded over the past few years. The authors, who confess to have added to the confusion by advancing a variety of different syntactic and semantic proposals, feel that it would be beneficial to pause for a second - to pause and look back to sort out what has been accomplished and what needs to be done." [AH92].

Twelve years after [AH92] was published the work on real time metric formalisms still does not converge towards an accepted model and a main formalism to argue about the model. The models and the formalisms continue to proliferate.

We believe that the original sin was in rejecting the Real Line as the obvious model for the flow of time in favor of sequential codings of real time bits of information. This was followed by abandoning the well developed classical logic and its model theory replacing it by various modifications of temporal logic.

Concerning the model: In a survey of the state of the art in 1992 [AH92] we read:

> "Any combination ... leads to one of sixteen possible formal semantics of real time systems".

All of these models are sequence models (see Definitions 11.1, 11.2). None is the real line itself. None of them is capable of expressing predicates that are not of finite variability and in particular none allows for treating finite variability predicates as a special case of general predicates. One of the sixteen semantics ("monotonic real time interval based systems") correspond to the intended real model at least as far as finite variability flows are concerned. Others like "weakly monotonic systems" do not seem to relate to any conceivable predicate (how do we understand the predicate p which is zero outside integer points and for each natural number n it alternates n times between 0 and 1?)

Nowhere did we find an attempt to explain the rational behind this choice. Take for example the very simple predicate P over the reals that is true for $0 \leq t \leq 2$ and false elsewhere. How can it

be represented in a point based semantics that makes it clear if P holds at 2 or not? In interval based semantics there are many representations of this predicate. One representation is the interval sequence $\sigma_1 = \langle I_1, P\rangle\langle I_2, P\rangle\langle I_3, \neg P\rangle$, where I_1 is interval $[0, 1.5]$, I_2 is interval $(1.5, 2]$ and I_3 is interval $(2, \infty)$; another representation is by interval sequence $\sigma_2 = \langle J_1, P\rangle\langle J_2, \neg P\rangle$, where J_1 is interval $[0, 2]$ and J_2 is interval $(2, \infty)$. There are some "logics" considered in the literature which distinguishes between these interval sequences, i.e.. they have a formula ψ which holds on the first sequence and does not hold on the second sequence.

A correct but unfortunate definition of "finite variability" in the reals [BKP85] led to a questionable definition in the rationales with the surprising consequence that

> "a TLR formula is satisfiable by a real model iff it is satisfiable by a rational model" [BKP85]

This result was later strengthened in [AFH96] where it was shown that even a metric $MITL$ formula cannot distinguish between the reals and the rationales ([AFH96], theorem 2.4.1.4). The property stating that a predicate p is downward closed, bounded and has no supremum is easily formalized as

$$p \wedge \Diamond\neg p \wedge Always(\neg p \rightarrow \Box\neg p) \wedge Always(p \rightarrow p\ until\ p) \wedge Always(\neg p \rightarrow \neg p\ since\ \neg p)$$

This formula is satisfiable in the rationals and is not satisfiable in the real line. In 15 years no one pointed out that proving this formula unsatisfiable in "sixteen possible formal semantics of rational time" does not reveal something deep about the rational line; it only reveals that our modelling method needs reconsideration.

Concerning the Logic: Adding metric to temporal logics with the sequence models in mind was not as easy as adding the $+1$ function symbol to a predicate logic. Constructs like "Freeze quantification", "Clock variables", "Half order logics", "explicit clock notations", "reset quantification", "position variables" and more were introduced without converging towards a universally accepted formalism. In fact only a handful of these formalisms made it to the finals: in "It's about time: Real time logics reviewed" [Hen98] the conclusion says:

> "In real time $TPTL$ is strictly more expressive than MTL which is strictly more expressive than the other three logics - $MITL$, $MITL_{0,\infty}$, and the future fragment of ECL. In point based real time $MITL$ is strictly more expressive than both $MITL_{0,\infty}$ and $ECTL$, which are equally expressive. In interval based real time all three logics are equally expressive."

This in fact is only half the list of candidates since "past temporal operators" may change the expressive power so that for each of the five formalisms there is one with past and one without:

How then are we to choose the correct logic without the yardstick of the standard model with the standard predicate logic?

In fact when adapted to the standard model all five logics coincide. They only differ on pairs "logic-model" that fail to obey basic integrity tests. "Point based real time" does not sustain axioms about logic and about the real line which we expect and almost take for granted:

$$\Diamond_1\Diamond_1 p \longleftrightarrow \Diamond_2 p$$

or simply

$$\Box\overleftarrow{\Diamond}_1 True.$$

Thus the state sequence $\langle 0, P\rangle, \langle 2, \neg P\rangle, \langle 4, \neg P\rangle, \langle 6, \neg P\rangle, \cdots$ satisfies at 0 the formula $\diamondsuit_2 \neg P$ but not $\diamondsuit_1 \diamondsuit_1 \neg P$, and in fact 0 does not even satisfy $\Box \overleftarrow{\diamondsuit}_1 True$. Moreover, there is no need for metric to see that point base (state sequence) models is inadequate for dense time. It is clear that $False\ Until\ True$ is not satisfiable over dense linear orders, but unfortunately it is valid over all kinds of state sequence models. The basic laws of temporal logics over state sequence models differ from the laws over dense time, hence state sequence models are inappropriate to model dense time.

It is difficult to understand why all the authors diverged from the straightforward approach of the real line as the model and the classical (first and second order) predicate logic as the logic. In the discussion above we tried to show that this was a costly choice that should, and can, be corrected.

References

[AD94] R. Alur, D. Dill A theory of timed Automata. Theoretical Computer Science 126 (1994), 183-235.

[AFH96] R. Alur, T. Feder, T.A. Henzinger. The Benefits of Relaxing Punctuality. Journal of the ACM 43 (1996) 116-146.

[AH92] R. Alur, T.A. Henzinger. Logics and Models of Real Time: a survey. In Real Time: Theory and Practice. Editors de Bakker et al. LNCS 600 (1992) 74-106.

[AH92a] R. Alur, T.A. Henzinger. Back to the future: toward theory of timed regular languages. In 33th FOCS, pp177-186, 1992.

[AH93] R. Alur, T.A. Henzinger. Real Time Logic: Complexity and Expressiveness. Information and Computation 104 (1993) 35-77.

[BKP85] H. Barringer, R. Kuiper, A. Pnueli. A really abstract concurrent model and its temporal logic. Proceedings of the 13th annual symposium on principles of programing languages (1986), 173-183.

[Bü60] J. Büchi. On a decision method in restricted second order arithmetic In *Proc. International Congress on Logic, Methodology and Philosophy of Science*, E. Nagel et al. eds, Stanford University Press, pp 1-11, 1960.

[BL95] A. Bouajjani and Y. Lakhnech. Temporal Logic +Timed Automata: Expressiveness and Decidability. CONCUR95, LNCS 962, pp. 531-547, 1995.

[BG85] J.P. Burges, Y. Gurevich. The Decision Problem for Temporal Logic. Notre Dame J. Formal Logic, 26 (1985) 115-128.

[Ehr61] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae*, 49:129–141, 1961.

[GHR94] D.M. Gabbay, I. Hodkinson, M. Reynolds. Temporal Logics volume 1. Clarendon Press, Oxford (1994).

[GPSS80] D.M. Gabbay, A. Pnueli, S. Shelah, J. Stavi. On the Temporal Analysis of Fairness. 7th ACM Symposium on Principles of Programming Languages. Las Vegas (1980) 163-173.

[Hen98] T.A. Henzinger It's about time: real-time logics reviewed. in Concur 98, Lecture Notes in Computer Science 1466, 1998.

[HRS98] T.A. Henzinger, J.F. Raskin, P.Y. Schobbens. The Regular Real Time Languages. 25th ICALP Colloquium (1998).

[HR99] Y. Hirshfeld and A. Rabinovich, A Framework for Decidable Metrical Logics. In Proc. 26th ICALP Colloquium, LNCS vol.1644, pp. 422-432, Springer Verlag, 1999.

[HR99a] Y. Hirshfeld and A. Rabinovich. Quantitative Temporal Logic. In Computer Science Logic 1999, LNCS vol. 1683, pp 172-187, Springer Verlag 1999.

[HR03] Y. Hirshfeld and A. Rabinovich, Timer formulas and decidable metric temporal logic. Technical report, Tel Aviv University (2003).

[Kamp68] H. Kamp. Tense Logic and the Theory of Linear Order. Ph.D. thesis, University of California L.A. (1968).

[MP93] Z. Manna, A. Pnueli. Models for reactivity. Acta informatica 30 (1993) 609-678.

[Ra69] M. O. Rabin, Decidability of second order theories and automata on infinite trees. In *Trans. Amer. Math. Soc.*,**141**,pp 1-35, 1969.

[Rab98] A. Rabinovich. On the Decidability of Continuous Time Specification Formalisms. In Journal of Logic and Computation, pp 669-678, 1998.

[Rab98a] A. Rabinovich. On the complexity of TL over the reals. Manuscript 1998.

[RSH98] J.F. Raskin, P.Y. Schobbens, T.A. Henzinger. Axioms for real-time logics. In CONCUR 98, Lecture Notes in Computer Science 1466, pp. 439-454, 1998.

[Re99] M. Reynolds. The complexity of the temporal logic with until over general linear time, manuscript 1999.

[She75] S. Shelah. The monadic theory of order. *Ann. of Math.*, **102**, pp 349-419, 1975.

[Trak99] B.A. Trakhtenbrot. Dynamic Systems and their interaction: Definitional Suggestions. Technical Report, Tel Aviv University, 1999.

[Wilke94] T. Wilke. Specifying Time State Sequences in Powerful Decidable Logics and Time Automata. In Formal Techniques in Real Time and Fault Tolerance Systems. LNCS 863 (1994), 694-715.