Chapter 8

Lecture 8

Advanced Topics in Graph Algorithms		
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8.1 Some interesting graph families characterized by intersection

8.1.1 Introduction

The topic of this lecture is intersection graphs. We shall focus mainly on F-graphs and tolerance graphs.

Definition Let S = $\{T_i | i \in I\}$ be a family of subsets of a set T. The *intersection graph* of S is $G = (S, E)$, where

- S = $\{T_i\}_j \in_I$
- E = {(i,j) $|T_i \cap T_j \neq \emptyset\}$.

It was proved (*Marczewski [1945]*) that if we allow S to be an arbitrary family, then we obtain all the undirected graphs. Usually we are interested in families of sets with specific topological properties. Moreover we should notice that some graphs were defined as intersection graphs (like line graphs), and others were defined by their properties, and only afterwards were proved to be an intersection graph (for example triangulated graphs, were originally studied as C_k-free graphs (k > 3) and only later characterized as intersection of subtrees).

8.1.2 Permutation graphs

Definition A *matching diagram* consists of n points, on each of two parallel lines, and n straight line segments matching the points. The intersection graph of the line segments is called a *permutation graph* (See Fig. 8.1). If the endpoints on the line are numbered 1,2,. . .n, then the endpoints on the other side of the line are numbered by a permutation on {1,2. . .,n} denoted σ.

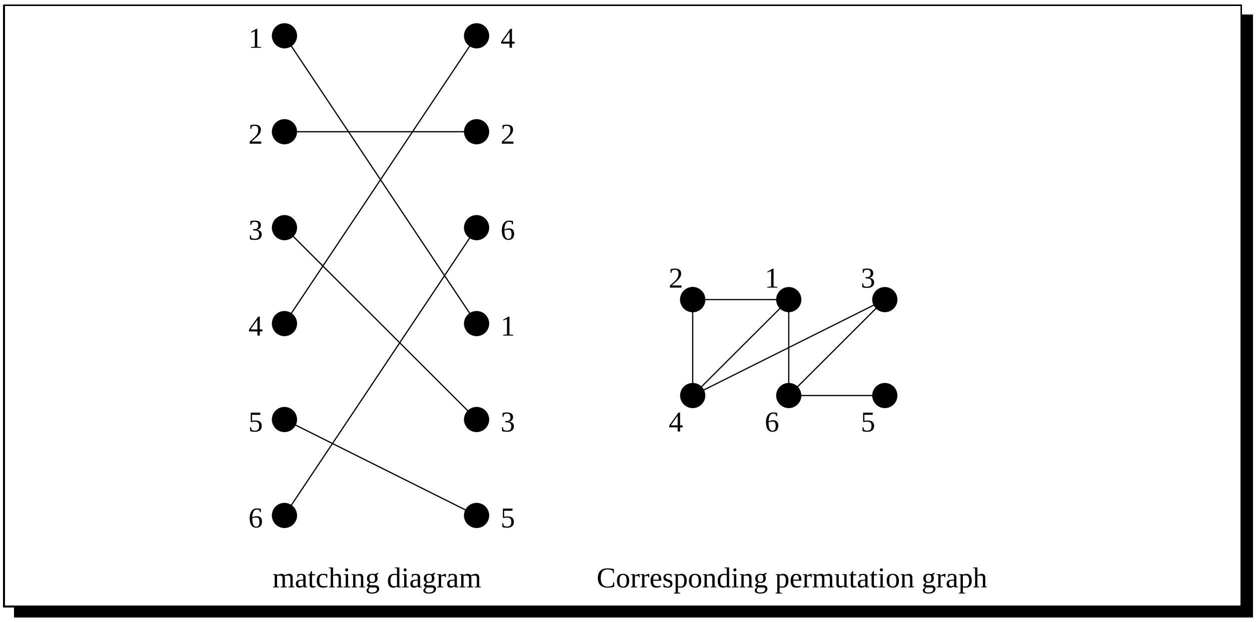

Figure 8.1: A permutation graph for the permutation [4,2,6,1,3,5]

Remark 8.1 *It is easy to build the permutation graph directly from the permutation.*
$\forall i, j\ [i < j \text{ and } \sigma(i) > \sigma(j)] \Leftrightarrow (i, j) \in E.$

Theorem 8.2 *A permutation graph $G = (V, E)$ is transitively orientable.*

Proof: Let σ be the permutation of $\{1, \ldots, n\}$ generating $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. For every $(i, j) \in E$, define F as follows :
$(i, j) \in F$ iff $i < j$.
We shall prove that $D = (V, F)$ is transitively orientable.
Let $(i, j) \in F, (j, k) \in F$, then by Remark 8.1, since $(i, j) \in E$, then $\sigma^{-1}(j) < \sigma^{-1}(i)$, and $\sigma^{-1}(k) < \sigma^{-1}(j)$. Hence $\sigma^{-1}(k) < \sigma^{-1}(i)$, and therefore $(i, k) \in F$.
which implies F is a transitive orientation (See fig. 8.2). ■

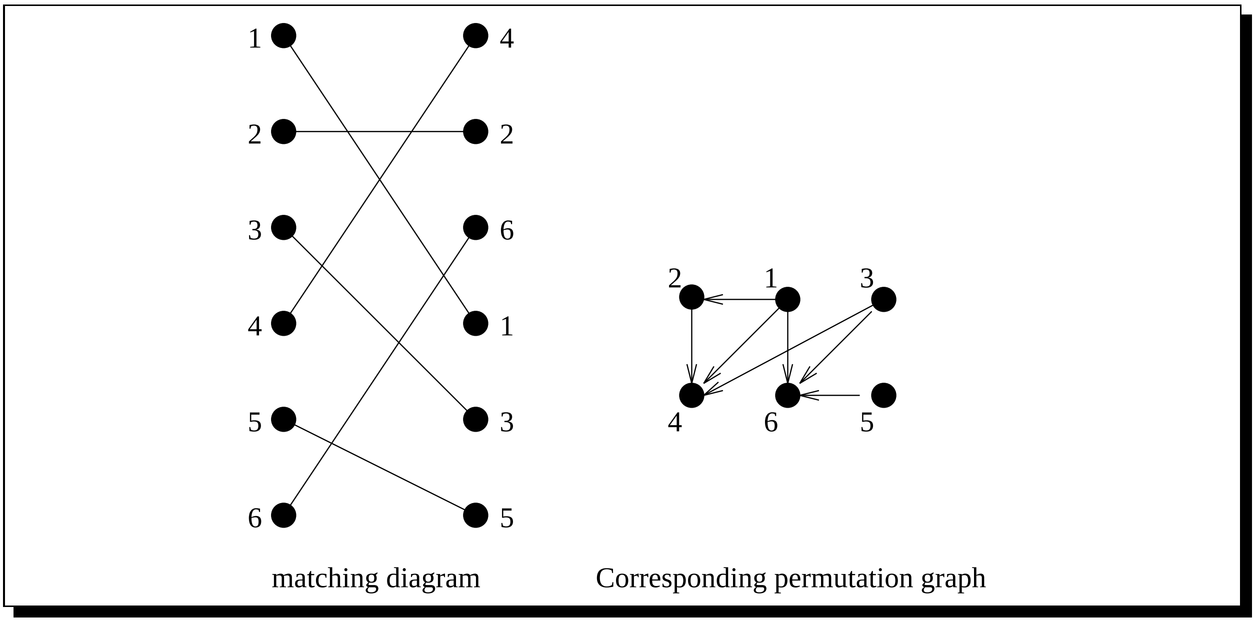

Figure 8.2: An orientation on a permutation graph

Definition If σ is a permutation, we say that $\bar{\sigma}$ is the opposite permutation, meaning that $\sigma(i) = \bar{\sigma}(n+1-i)$. For example $\sigma = [5\ 6\ 7\ 2\ 3\ 8\ 1\ 9\ 4]$,$\bar{\sigma} = [4\ 9\ 1\ 8\ 3\ 2\ 7\ 6\ 5]$.

Theorem 8.3 *If G is a permutation graph generated by σ, then $\bar{G}$ is also a permutation graph, generated by $\bar{\sigma}$.*

Proof: We shall prove that the graph $H = (V, F)$ generated by $\bar{\sigma}$ is $\bar{G}$.

For every i,j$\in$V, $(i, j) \in E$ iff (i-j)$(\sigma^{-1}(i) - \sigma^{-1}(j)) < 0$. But since $\bar{\sigma}^{-1}(i) < \bar{\sigma}^{-1}(j)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$ and (i,j)$\in$F iff (i-j)$(\bar{\sigma}^{-1}(i) - \bar{\sigma}^{-1}(j))$ >0 , we obtain that (i,j)$\in$E $\Leftrightarrow$ (i,j)$\notin$F, so H=$\bar{G}$. ∎

Theorem 8.4 *(Pnueli Lempel Even 1971) G is a permutation graph iff both G and $\bar{G}$ are transitively orientable.*

Proof:

$\Rightarrow$ Trivial from the previous two theorems.

$\Leftarrow$ Wasn't proved in class.

∎

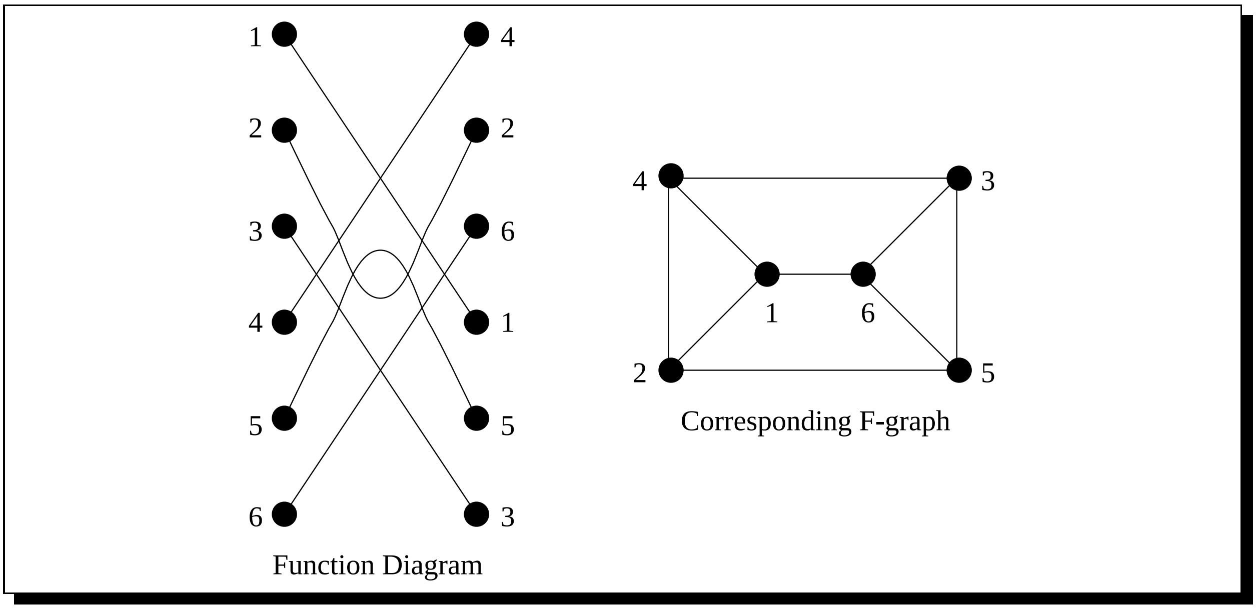

Figure 8.3: An F-graph

8.1.3 F-Graphs

Definition An *F-diagram* consists of n points, on each of two parallel lines, and n continuous function curves ending in the points. The intersection graph of the functions segments is called a function graph, or an *F-graph* (See Fig. 8.3). Note the requirement that each curve segment is a function (See Fig. 8.4 for negative example).

"The air controllers strike"

Given two collections of cities lying on two parallel lines, and a collection of flight paths between them, assign an altitude to each flight so that no crashes occur. It is easy to see that if we take the intersection graph of the paths, we obtain an F-graph (assuming that no flight is going backwards). We should assign different altitude only for those flights that intersect. Therefore we can consider each altitude to be a color, and we should solve the coloring problem for that F-graph (See Fig. 8.5). Note that if all flight paths are straight lines, then the resulting graph is a permutation graph.

A composition of permutation diagram.

We can think of F-graph as if it was constructed from few partial orders (each of them is a permutation on the vertices). Therefore we can construct the F-graph from the composition of these partial orders. (See Fig. 8.6).

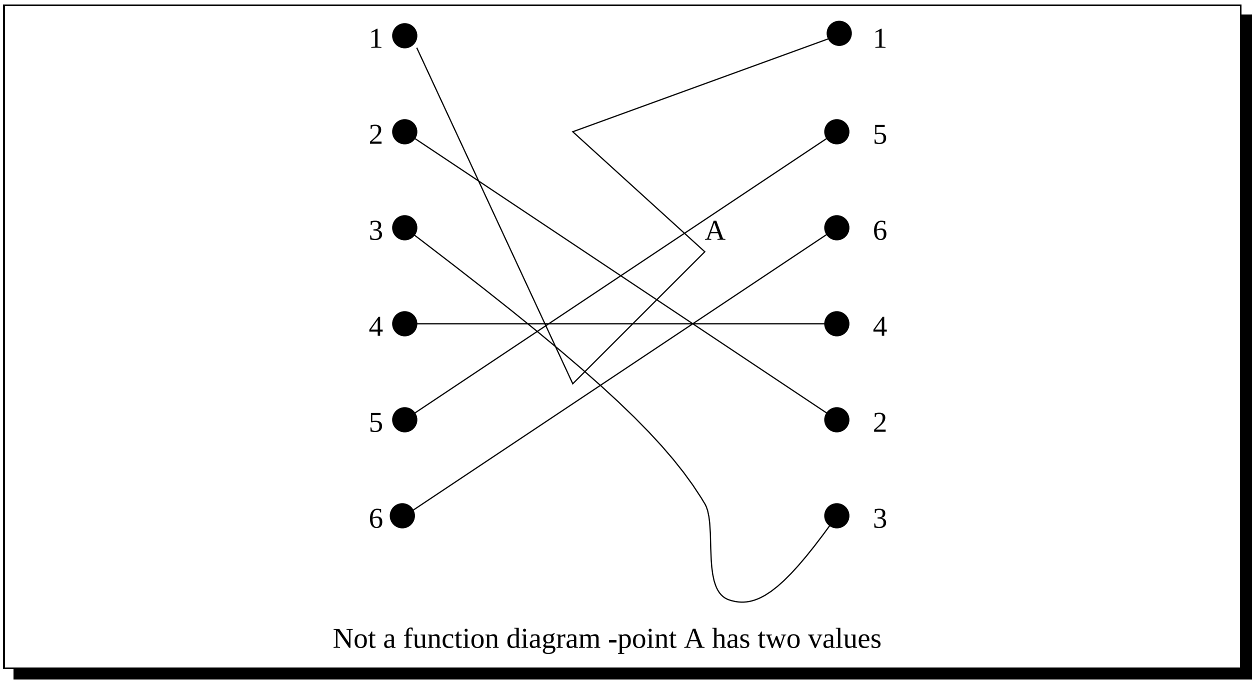

Figure 8.4: Not an F-graph,point A have two values of the function.

Theorem 8.5 *(Golumbic Rotem Urrutia 1982)*
Let G be an undirected graph. The following statements are equivalent.

(1) G is an F-graph.

(2) G is the intersection graph of a composition of permutation diagrams.

(3) $\bar{G}$ is transitively orientable.

Proof:

(2) $\Rightarrow$ (1) Trivial, each piecewise-linear path (connecting the point numbered i in all the orders), is a continuous function. Therefore it is an F-Graph.

(1) $\Rightarrow$ (3) We shall define $H = (V, F)$ an orientation of $\bar{G}$ as follows : (i,j) $\in$F if the i-th path in the diagram is completely below the j-th path. More formally :

(i,j)$\in$F if $\forall$ x $f_i(x) < f_j(x)$ and (j,i) if $\forall$ x $f_j(x) < f_i(x)$, otherwise the edge is not in F.

If $(i, j) \in E$, than obviously the function $f_i(x) - f_j(x)$ changes sign at least once. Therefore (i,j) $\notin F$, and vice versa. So $F = \bar{E}$. We should now prove that F is transitive orientation for $\bar{G}$.

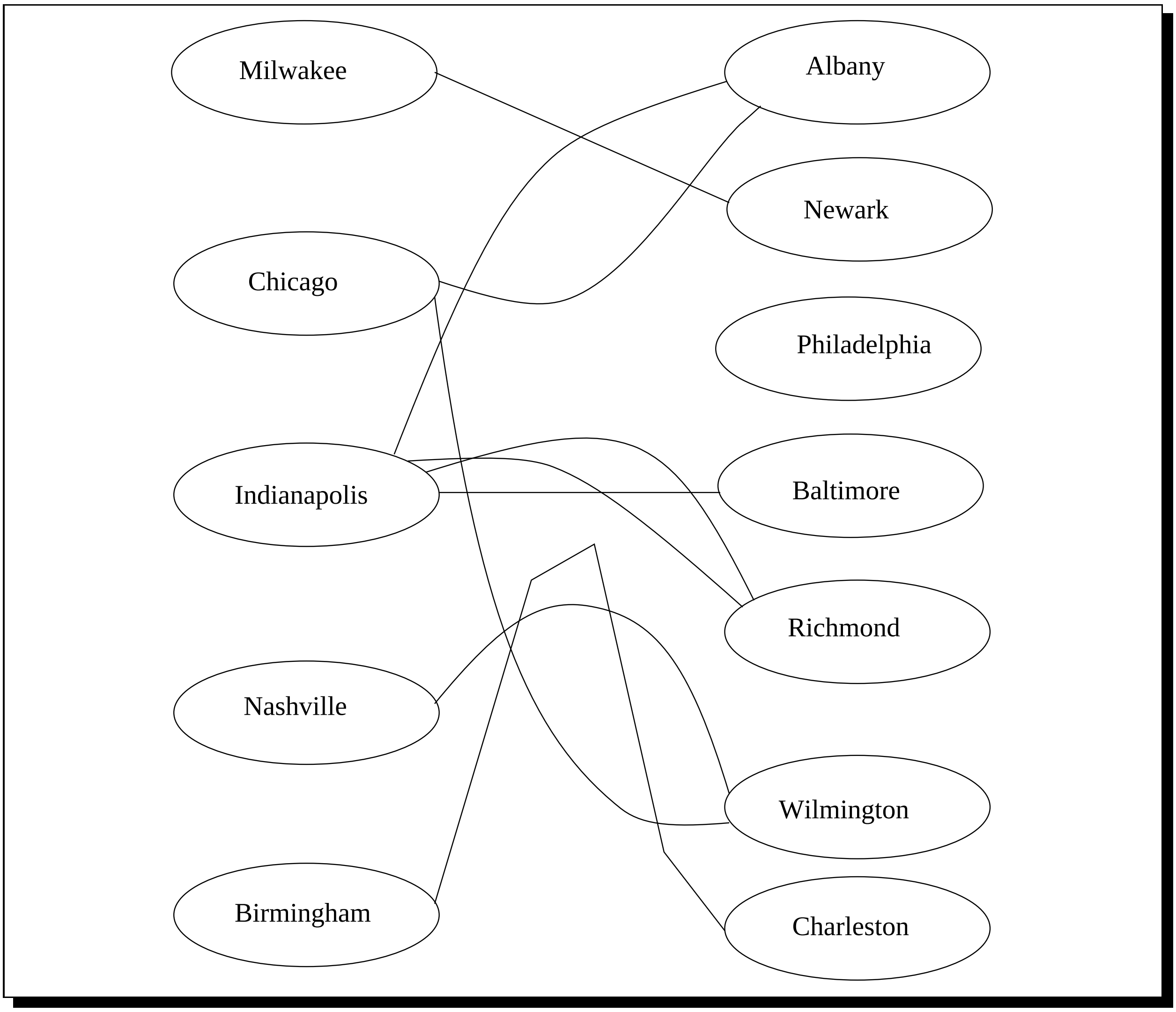

Figure 8.5: The air controllers strike problem.

If $(i,j) \in F$, and $(j,k) \in F$, then : $\forall x f_i(x) < f_j(x) < f_k(x)$.
$\Rightarrow (i,k) \in F$.

(3) $\Rightarrow$ (2) Let F be a transitive orientation of $\bar{G}$. We can choose $[l_1, l_2, \dots l_k]$ to be a realizer of F. For each line order l_i we assign the points of its permutation in the corresponding order on the i-th horizontal line (See fig. 8.6). For i=1,2,...,n connect all points corresponding to i by straight line segment. The resulting structure is a function diagram.

Now, by the definition of the realizer, (i,j)∈F iff for every order l_k i,j appear in the same order (i.e. either for all k $i <_{l_k} j$ or for all k $j <_{l_k} i$). Hence (i,j)∉F iff the paths of i and j intersect. Hence G is an F-graph.

■

8.1.4 Tolerance graphs

Definition Let $I = \{I_1, I_2, \dots I_n\}$ be a collection of line segments, and let $T = \{t_1, t_2, \dots t_n\}$ a set of positive numbers. The *tolerance graph* $G = (V, E)$ of (I,T) is defined as follows :
$V = \{v_1, \dots, v_n\}$
$E = \{(i,j) | |I_i \cap I_j| \geq min(t_i, t_j)\}$ (See Fig. 8.7)

Interval graphs as a subset of tolerance graphs.

The family of tolerance graphs is a generalization of the interval graphs. They are usually used for more general scheduling problems, where we allow things to happen simultaneously.

Given an interval graph $G = (V, E)$ whose realization is $I = \{I_1, I_2, \dots, I_n\}$, we can construct an isomorphic tolerance graph in the following manner :
$T = (V, E)$ the tolerance graph of (I, X):
define $X(i,j) = |I_i \cap I_j|$, if $I_i \cap I_j \neq \emptyset$
$C = min\{X_(i,j) | (i,j) \in E\}$ Now let I=$\{I_1, I_2, \dots, I_n\}$ and T=$\{t_1, t_2, \dots, t_n\}$ where $t_i = C/2$ for all i. We claim that G is the tolerance graph of (I,T). Two line segments intersect in the original interval graph, iff the intersection length is bigger than C/2, therefore the edge sets of the two graphs coincide.

The following lemma will be used in the next theorem :

Definition G=(V,E) is an *interval containment* graph if there exists a set of intervals $\{I_u\}_{u \in V}$ such that (u,v)∈E⇔ $[I_u isin I_v$ or vice versa].

Lemma 8.6 *G is a permutation graph if and only if it is a interval containment graph.*

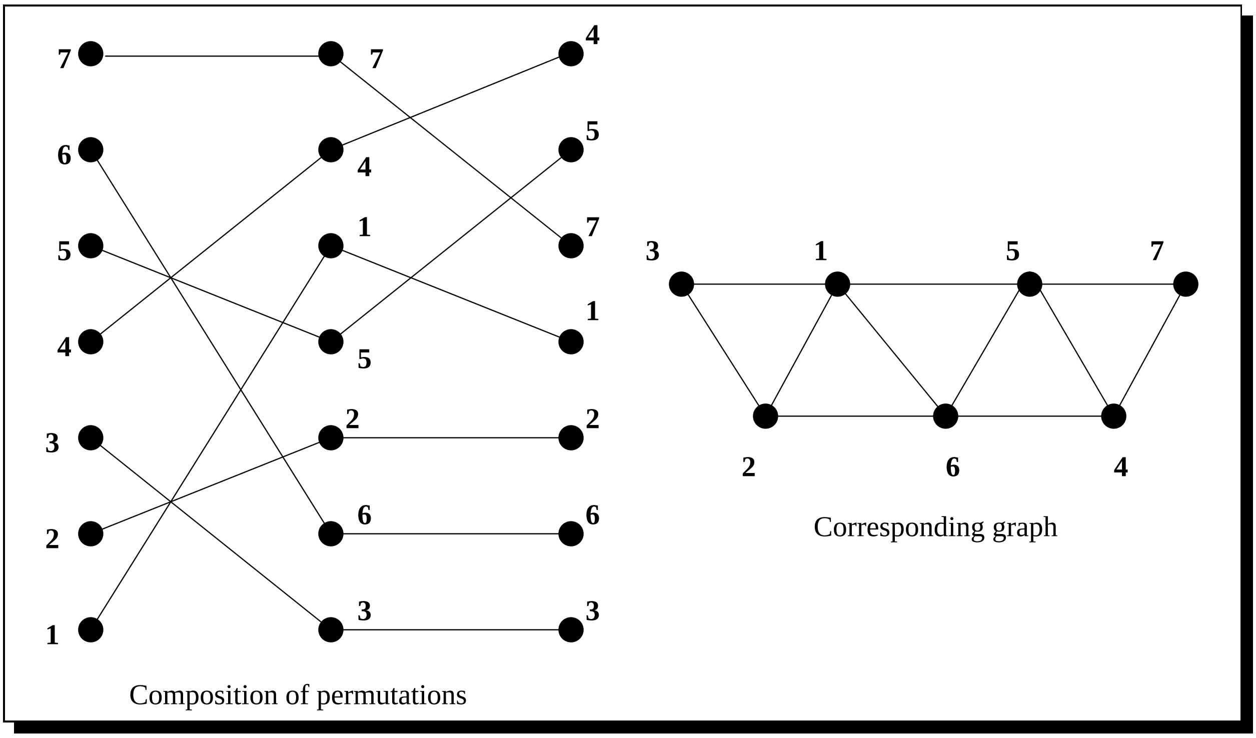

Figure 8.6: The composition of permutation diagrams.

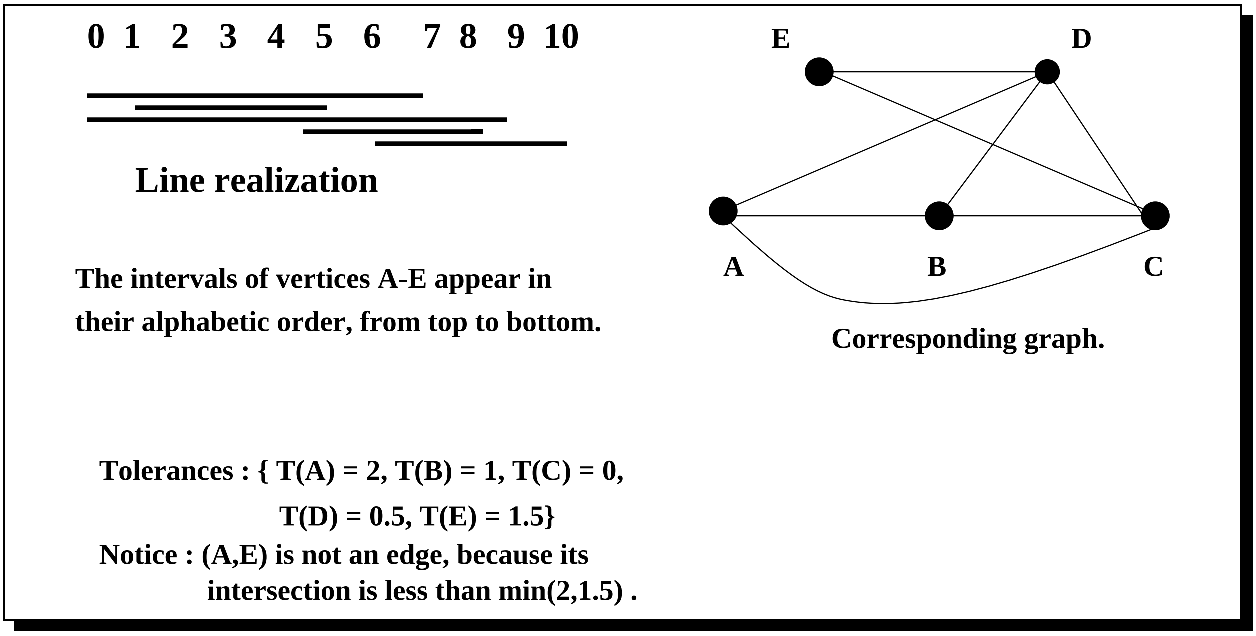

Figure 8.7: A tolerance graph.

Proof: Both sides wasn't proved in class. ■

Theorem 8.7 *G is a tolerance graph with $t_i = |I_i|$, if and only if G is a permutation graph.*

Proof:

$\Rightarrow$ The proof is based on a the previous lemma. So all we have to do is to prove that G is an interval containment graph.

Let G be a tolerance graph with t_i defined as above. If $(i, j) \in E$ then either I_j contains I_i, or vice versa. Using the previous theorem, our theorem is proved.

$\Leftarrow$ If G is a permutation graph, than it is a line containment graph, therefore we can realize it with a tolerance graph with tolerances defined as above.

■

8.1.5 Bounded and unbounded tolerance graphs.

If for a vertex i $t_i > |I_i|$, than we can enlarge t_i as much as we want, without affecting the graph. Any interval which intersects I_i will have a common intersection with I_i not longer than t_i, so in this case an edge (i,j) will be in the graph if and only if $|I_i \cap I_j| \geq t_j$. V_i will be called a vertex with *unbounded tolerance.* If G is a tolerance graph, in which no v_i is of unbounded tolerance, we call G a *bounded tolerance graph.*

Theorem 8.8 *Let G be a bounded tolerance graph, then is an F-graph.*

Proof: Wasn't proved in class. ■