ADVANCED NUMBER THEORY 2008
ASSIGNMENT 2: THE KRONECKER SYMBOL

Due date: Wednesday, June 11, 2008

Recall from the basic number theory course that for $m > 0$ odd, the Jacobi symbol $\left(\frac{a}{m}\right)$ is defined in terms of the Legendre symbol by writing $m = \prod p_i$ as a product of odd primes and then setting $\left(\frac{a}{m}\right) := \prod_i \left(\frac{a}{p}\right)$. The law of quadratic reciprocity states that for $m > 0$ odd,

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m-1}{2}}, \quad \left(\frac{2}{m}\right) = \begin{cases} 1, & m = \pm 1 \mod 8 \\ -1, & m = \pm 3 \mod 8 \end{cases}$$

and that for $m > 0$, $n > 0$ coprime and odd we have

$$\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right) (-1)^{\frac{m-1}{2}\frac{n-1}{2}}$$

The ***Kronecker symbol*** $\left(\frac{d}{m}\right)$ is defined for and $m > 0$ and $d = 0, 1 \mod 4$ not a perfect square, by requiring

$$\left(\frac{d}{2}\right) = \begin{cases} 1 & d = 1 \mod 8 \\ -1 & d = 5 \mod 8; \end{cases}$$

for an odd prime p, $\left(\frac{d}{p}\right) = 0$ if $p \mid d$ and $\left(\frac{d}{p}\right)$ is the Legendre symbol for $p \nmid d$; and for $m = \prod_i p_i$ set $\left(\frac{d}{m}\right) = \prod_i \left(\frac{d}{p_i}\right)$. In particular for $m > 0$ odd this is just the Jacobi symbol. Show that

1. a) $\left(\frac{d}{2}\right) = \left(\frac{2}{|d|}\right)$, d odd.
b) For $m, n > 0$, $\left(\frac{d}{mn}\right) = \left(\frac{d}{m}\right)\left(\frac{d}{n}\right)$

2. If $m > 0$ is coprime to d then $\left(\frac{d}{m}\right) = \left(\frac{m}{|d|}\right)$ d odd.

3. If $d = 2^b d_1$ with d_1 odd and $b > 0$ then

$$\left(\frac{d}{m}\right) = \left(\frac{2}{m}\right)^b (-1)^{\frac{d_1-1}{2}\frac{m-1}{2}} \left(\frac{m}{|d_1|}\right)$$

4. Define $\chi_d(m) := \left(\frac{d}{m}\right)$ for $m > 0$. Show that $\chi_d(m) = \chi_d(n)$ if $m \equiv n \mod |d|$. Conclude that χ_d is a real Dirichlet character mod $|d|$.