ADVANCED NUMBER THEORY 2008
PROF. ZEÉV RUDNICK
TAKE-HOME EXAM

Due date: Wednesday, July 30, 2008.

Please hand it in to me either in person, in my mailbox or via email.
You may not consult with anyone except myself !

I. Find all equivalence classes of integer solutions of the following equations $f(x, y) = k$, where equivalence refers to the action of the isometry group $\mathrm{Aut}(f) \subset SL_2(\mathbb{Z})$ on solutions:

1) $x^2 - 11y^2 = 5$

2) $3x^2 + 2xy - 3y^2 = 2$

3) $x^2 - 2y^2 = -7$.

II. Let $d < 0$ be a discriminant, and $h(d)$ the number of equivalence classes of positive definite forms of discriminant d.

1) For an integer $f > 1$, show that the class numbers $h(d)$ and $h(df^2)$ are related by

$$\frac{h(df^2)}{h(d)} = \frac{2}{w_d} f \prod_{p|f} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right)$$

the product over all primes dividing f, and where $w_{-3} = 6$, $w_{-4} = 4$, and $w_d = 2$ if $d < -4$.

2) Deduce that the ratio $h(df^2)/h(d)$ is an integer.

III. Riemann's ζ-function is given for $Re(s) > 1$ as the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ .$$

In particular $\zeta(s) > 0$ for real $s > 1$. In class we proved that it has an analytic continuation to $Re(s) > 0$. Show that $\zeta(s) < 0$ for real $0 < s < 1$.