ADVANCED NUMBER THEORY 2012
ASSIGNMENT 4
DUE DATE: THURSDAY, JUNE 7, 2012

The following exercises are to gain facility with using Stirling's formula, which says that

$$\log\Gamma(s) = (s-\frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + O(\frac{1}{|s|})$$

as $|s|\to\infty$ in a sector $|\arg s| < \pi-\delta$, and that in that sector

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(\frac{1}{|s|})$$

1. Show that the binomial coefficients satisfy

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \qquad \text{as } n\to\infty$$

2. Show that for $s=\frac{1}{2}+it$, t real, as $t\to+\infty$,

$$\frac{1}{\pi}\mathrm{Im}\log\left(\pi^{-s/2}\Gamma(\frac{s}{2})\right) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} - \frac{1}{8} + O(\frac{1}{t})$$

3. Using the functional equation for $\zeta(s)$ and Stirling's formula, show that for $s=\sigma+it$ with $\sigma=\mathrm{Re}(s)<0$,

$$|\zeta(s)| \ll (\frac{t}{2\pi})^{\frac{1}{2}-\sigma}$$

as $t\to+\infty$ ($\sigma<0$ is fixed), the implied constant depending on σ.