ADVANCED NUMBER THEORY 2012
TAKE-HOME EXAM
DUE DATE: JULY 19, 2012

1. a) Let $\phi(n)$ be Euler's totient function, giving the number of invertible residues modulo n. Show that the Dirichlet series attached to ϕ satisfies

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad \mathrm{Re}(s) > 2$$

b) Let $d(n) = \sum_{d|n} 1$ be the number of divisors of n. Show that

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}, \quad \mathrm{Re}(s) > 1$$

2. Let $\omega(n)$ be the number of distinct prime divisors of n, e.g. $\omega(12) = 2$.

a) Show that $\omega(n) \leq \log_2 n$.

b) Let $n_K = \prod_{p \leq K} p$ (the product of all primes $\leq K$). Show that

$$\omega(n_K) \sim \frac{\log n_K}{\log \log n_K}, \quad K \gg 1$$

c) Show that the mean value of $\omega(n)$ satisfies

$$\frac{1}{N} \sum_{n \leq N} \omega(n) \sim \log \log N, \quad N \to \infty$$

Hint: Use $\omega(n) = \sum_{p|n} 1$ and switch order of summation.

3. Let $Q > 1$, and for χ a Dirichlet character modulo Q let $L(s, \chi)$ be the Dirichlet L-function corresponding to χ. Show that for $s = \sigma + it$, $\sigma = \mathrm{Re}(s) > 1$, and Q prime,

$$\frac{1}{\phi(Q)} \sum_{\chi \mod Q} \left| L(s, \chi) \right|^2 = \zeta(2\sigma) + o(1)$$

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Let $w \geq 0$ be a weight function, satisfying $\int_{-\infty}^{\infty} w(x)dx = 1$, so that its Fourier transform $\widehat{w}(x) = \int_{-\infty}^{\infty} w(t)e^{-2\pi ixt}dt$ is smooth and supported in $[-1,1]$, and set $\mathbb{E}_{w,T}\Big\{F(t)\Big\} := \int_{-\infty}^{\infty} F(t)w(\frac{t}{T})\frac{dt}{T}$.

4. Fix $\sigma > 1$. Show that for t real,

$$\lim_{T\to\infty} \mathbb{E}_{w,T}\Big\{ |\zeta(\sigma+it)|^2 \Big\} = \zeta(2\sigma)$$

Hint: If $m \neq n$ are positive integers, show that $|\log\frac{m}{n}| > 1/(2\min(m,n))$.

5. Let

$$R_X(t) = -\frac{1}{\pi}\sum_{p\leq X}\frac{\sin(t\log p)}{\sqrt{p}}$$

Show that for $T^\delta < X < T^{1/2-\delta}$ $(0 < \delta < 1/4$ small), the 4-th moment of $R_X(t)$ is asymptotically given by

$$\mathbb{E}_{w,T}\Big\{(R_X(t))^4\Big\} \sim 3(\frac{1}{2\pi^2}\log\log T)^2, \quad T\to\infty$$