TOPICS IN NUMBER THEORY 2013
ASSIGNMENT 2
DUE DATE: APRIL 9, 2013

1. For $0 \neq f \in \mathbb{F}_q[x]$, let $\tau(f)$ be the number of monic divisors of f. Show that mean value of $\tau(f)$ over monic polynomials of degree n in $\mathbb{F}_q[x]$ is $n+1$, that is show that

$$\frac{1}{q^n} \sum_{\substack{\deg f = n \\ f \text{ monic}}} \tau(f) = n+1$$

Hint: We showed that $\sum \tau(f)|f|^{-s} = \zeta_q(s)^2$.

2. Euler's totient function for $\mathbb{F}_q[x]$ is $\Phi(f) = \#\Big(\mathbb{F}_q[x]/(f)\Big)^\times$, the number of invertible residues modulo f ($f \neq 0$), equivalently the number of polynomials A with $\deg A < \deg f$ co-prime to f (if $\deg f = 0$ then we set $\Phi(f) = 1$). Show that

$$\sum_{d|f} \Phi(d) = |f|$$

the sum over monic divisors of f.

3. Show that

$$\sum_{0 \neq f \in \mathbb{F}_q[x] \text{ monic}} \frac{\Phi(f)}{|f|^s} = \frac{\zeta_q(s-1)}{\zeta_q(s)}, \quad \mathrm{Re}(s) > 2$$

Deduce that

$$\sum_{\substack{\deg f = n \\ f \text{ monic}}} \Phi(f) = \frac{q^{2n}}{\zeta_q(2)}, \quad n > 0$$

4. Show that

$$\sum_{0 \neq f \in \mathbb{F}_q[x] \text{ monic}} \frac{\tau(f^2)}{|f|^s} = \frac{\zeta_q(s)^3}{\zeta_q(2s)}, \quad \mathrm{Re}(s) > 1$$

5. Find the mean value of $\tau(f^2)$ over monic polynomials of degree n in $\mathbb{F}_q[x]$, that is

$$\frac{1}{q^n} \sum_{\substack{\deg f = n \\ f \text{ monic}}} \tau(f^2)$$

Hint: It is of the form $\alpha_q n^2 + \beta_q n + \gamma_q$.