1. Arithmetic in \( \mathbb{F}_q[t] \)

In this section \( \mathbb{F}_q \) denotes a field of \( q \) elements.

**Exercise 1.** Define the Mobius function for monic polynomials by 
\[
\mu(f) = (-1)^r \quad \text{if } f \text{ is a product of } r \text{ distinct monic irreducibles.}
\]
Show that 
\[
\sum_{\deg f = n} \mu(f) = 0, \quad n \geq 2
\]
the sum over monic polynomials of degree \( n \). For \( n = 1 \), show that the corresponding sum equals \(-q\).

**Hint:** Compute the generating series \( \sum f \mu(f)/||f||^s \).

**Exercise 2.** Show that the number of squarefree monic polynomials \( f \in \mathbb{F}_q[t] \) of degree \( n \geq 2 \) is 
\[
q^n (1 - q^{-1}) = q^n/\zeta_q(2)
\]

**Exercise 3.** Given distinct elements \( \alpha_1, \ldots, \alpha_r \in \mathbb{F}_q \), and \( \beta_1, \ldots, \beta_r \in \mathbb{F}_q \), show that there is a polynomial \( c(t) \in \mathbb{F}_q[t] \) so that \( c(\alpha_j) = \beta_j \).

Show that \( c(t) \) is coprime to \( \prod_{j=1}^r (t - \alpha_j) \) if and only if all \( \beta_j \neq 0 \).

**Exercise 4.** Show that given distinct elements \( \alpha_1, \ldots, \alpha_r \in \mathbb{F}_q \), and nonzero \( \beta_1, \ldots, \beta_r \in \mathbb{F}_q^\times \), there are infinitely many monic irreducible polynomials \( P \in \mathbb{F}_q[x] \) for which \( P(\alpha_j) = \beta_j, j = 1, \ldots, r \). (Hint: First try the case \( P(0) = 1 \).)

2. Discriminants and Class Numbers

**Exercise 5.** We computed the mean value of the normalized class numbers \( h(D)/|D|^{1/2} \) for negative discriminants. Deduce that 
\[
\sum_{0 < D < X \atop D \equiv 0, 1 \mod 4} h(D) = \frac{\pi}{18 \zeta(3)} X^{3/2} + O(X \log X).
\]

The goal of the following exercises is to compute the density of fundamental discriminants. Recall that a fundamental discriminant is an integer of the form \( D = 1 \mod 4 \) which is squarefree, or \( D = 4d \) with \( d \) squarefree and \( d = 2, 3 \mod 4 \). These are the discriminants of quadratic
fields. We denote by $D_{-1}(x)$ the set of odd negative fundamental discriminants with $0 < -D < x$, and by $D_{-0}(x)$ the set of even negative fundamental discriminants with $0 < -D < x$.

Let $\chi_0$ be the principal character mod 4, that is $\chi_0(n) = 1$ if $n$ is odd, and $\chi_0(n) = 0$ if $n$ is even, and $\chi_1$ the nonprincipal character mod 4, so $\chi_1(n) = +1$ if $n = 1 \mod 4$, and $\chi_1(n) = -1$ if $n = 3 \mod 4$, and $\chi_1(n) = 0$ if $n$ is even. Denote by $\mu$ the Mobius function.

Exercise 6. Show that for $s > 1$,
\[
\sum_{n=1}^{\infty} \frac{\mu(n)\chi_0(n)}{n^s} = \left(1 - \frac{1}{2^s}\right)^{-1} \frac{1}{\zeta(s)}.
\]

Exercise 7. Show that as $Y \to \infty$,
\[
\sum_{n \leq Y} \mu^2(n)\chi_1(n) = O\left(\sqrt{Y}\right).
\]

Exercise 8. Show that as $Y \to \infty$,
\[
\sum_{n \leq Y} \mu^2(n)\chi_0(n) = \frac{2}{3\zeta(2)}Y + O\left(\sqrt{Y}\right).
\]

Remark: We saw that the number of squarefree integers $n \leq Y$ is asymptotic to $Y/\zeta(2)$. This is similar except that there is a congruence condition involved. Why is $2/3$ the correct factor?

Exercise 9. Show that as $x \to \infty$,
\[
\#D_{-1}(x) = \frac{x}{3\zeta(2)} + O(x^{1/2}).
\]

The case of even discriminants is similar, just a bit more complicated, so is not required, but is listed to complete the picture:
\[
\#D_{-0}(x) = \frac{x}{6\zeta(2)} + O(x^{1/2}).
\]

What does this say about the density of all fundamental discriminants? Why is this a reasonable answer?