Our goal is to explain the analytic continuation and functional equation of the Riemann zeta function.

1. The Poisson summation formula

1.1. The Fourier transform. The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of smooth functions on the real line which are rapidly decaying together with their derivatives. It contains the subspace of compactly supported functions $C^\infty_c(\mathbb{R})$. Other examples are the Gaussian $e^{-kx^2}$, $k > 0$.

Note that the $L^2$ inner product

\[ \langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \]

is well defined on $\mathcal{S}(\mathbb{R})$, turning it into an infinite-dimensional inner product space. It is however not complete; its completion is $L^2(\mathbb{R})$.

The Fourier transform of $f \in \mathcal{S}(\mathbb{R})$ is defined by

\[ \hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{-2\pi ixy}dy \]

This is in fact defined for any $f \in L^1(\mathbb{R})$. Integration by parts shows that $\hat{\cdot} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ takes Schwartz functions to Schwartz functions; but it does not preserve $C^\infty_c(\mathbb{R})$, in fact the Fourier transform
of \( f \in C^\infty_c(\mathbb{R}) \) is entire, so cannot be compactly supported unless it is identically zero.

An important example is the Fourier transform of a Gaussian: For \( f_1(x) = e^{-\pi x^2} \) we have
\[
\hat{f}_1 = f_1.
\]
The Fourier transform commutes with dilations: Define the dilation operator for \( t > 0 \) by
\[
\lambda_t f(x) = f(tx),
\]
then
\[
\hat{\lambda_t f}(x) = t(\lambda_{1/t} f)(x) = \frac{1}{t} f \left( \frac{x}{t} \right).
\]
So for instance, taking \( f_t(x) = \lambda_t f_1(x) = e^{-\pi t x^2} \), we find
\[
\hat{f}_t = \frac{1}{\sqrt{t}} f_{1/t}.
\]
The Fourier transform is an isometry:
\[
||\hat{f}||_2 = ||f||_2, \quad f \in S(\mathbb{R}).
\]
Therefore the Fourier transform can be extended to a unique isometry of \( L^2(\mathbb{R}) \).

Fourier’s inversion formula says that the Fourier transform is essentially an involution: for \( f \in S(\mathbb{R}) \),
\[
\hat{\hat{f}}(x) = f(-x).
\]
The Poisson summation formula says that the sum of \( f \) over the lattice \( \mathbb{Z} \subset \mathbb{R} \) equals to the sum of its Fourier transform:
\[
\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \quad f \in S(\mathbb{R}).
\]

1.2. **The theta function.** We define the theta function for \( t > 0 \) by
\[
\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi t n^2}
\]
This sum converges. As \( t \to \infty \), we have \( \theta(t) \to 1 \).

**Theorem 1.1.** The functional equation of \( \theta \) is
\[
\theta \left( \frac{1}{t} \right) = \sqrt{t} \theta(t).
\]

**Proof.** We use the Poisson summation formula for the dilated Gaussian \( f_t \). Since \( \hat{f}_t = t^{-1/2} f_{1/t} \) we obtain
\[
\theta(t) := \sum_{n \in \mathbb{Z}} f_t(n) = \sum_{m \in \mathbb{Z}} \hat{f}_t(m) = \frac{1}{\sqrt{t}} \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t} = \frac{1}{\sqrt{t}} \theta \left( \frac{1}{t} \right)
\]
as claimed. □

Corollary 1.2. As $t \to 0^+$,

$$\theta(t) \sim \frac{1}{\sqrt{t}}.$$  

2. Analytic continuation and functional equation of $\zeta(s)$

We saw that $\zeta(s)$, initially defined for $\operatorname{Re}(s) > 1$, admits an analytic continuation to $\operatorname{Re}(s) > 0$ except for a simple pole at $s = 1$, having residue 1 there. We now improve this.

2.1. The Gamma function. Recall the Gamma function, defined for $\operatorname{Re}(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt.$$  

Using integration by parts, we see the functional equation (recurrence formula)

$$\Gamma(s + 1) = s\Gamma(s).$$  

In particular, we see that for any positive integer $n \geq 1$, we have $\Gamma(n + 1) = n!$.

The functional equation allows us to extend $\Gamma(s)$ analytically to all of $\mathbb{C}$, except for simple poles at $s = 0, -1, -2, \ldots$, where the residue is

$$\operatorname{Res}_{s=n} \Gamma(s) = \frac{(-1)^n}{n!}.$$  

Stirling’s formula says that in a sector $-\pi + \delta < \arg(s) < \pi - \delta$, $|s| \to \infty$, $\delta > 0$ fixed,

$$\log \Gamma(s) = (s - \frac{1}{2})\log s - s + C + O\left(\frac{1}{|s|}\right), \quad C = \frac{1}{2}\log 2\pi.$$  

Other noteworthy properties, are Euler’s reflection formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

which implies that $\Gamma(s) \neq 0$ is never vanishing, and Legendre’s duplication formula

$$\Gamma(s)\Gamma(s + \frac{1}{2}) = 2^{1-2s}\pi^{-1/2}\Gamma(2s).$$
2.2. The completed zeta function. We define for \( \text{Re}(s) > 1 \)

\[
\zeta^*(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s)
\]

**Theorem 2.1.** \( \zeta^*(s) \) has an analytic continuation to all \( \mathbb{C} \) except for simple poles at \( s = 0, 1 \). It satisfies the functional equation

\[
\zeta^*(s) = \zeta^*(1 - s).
\]

**Proof.** We start with the integral representation

\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) n^{-s} = \int_0^\infty e^{-\pi t n^2 t^{s/2}} \frac{dt}{t}, \quad \text{Re}(s) > 0
\]

which gives, for \( \text{Re}(s) > 1 \)

\[
\zeta^*(s) = \int_0^\infty \omega(t) t^{s/2} \frac{dt}{t}
\]

where

\[
\omega(t) = \frac{\theta(t) - 1}{2}.
\]

From the asymptotics of \( \theta(t) \) we obtain

\[
\omega(t) = O \left( e^{-\pi t} \right), \quad t \to +\infty
\]

and

\[
\omega(t) \sim \frac{1}{2\sqrt{t}}, \quad t \to 0^+.
\]

Thus the integral for \( \zeta^*(s) \) is absolutely convergent for \( \text{Re}(s) > 1 \).

The functional equation of \( \theta(t) \) implies

\[
\omega \left( \frac{1}{t} \right) = t^{1/2} \omega(t) + \frac{t^{1/2} - 1}{2}.
\]

We use this to cut the integral for \( \zeta^*(s) \) into two parts

\[
\zeta^*(s) = \int_0^1 \omega(t) t^{s/2} \frac{dt}{t} + \int_1^\infty \omega(t) t^{s/2} \frac{dt}{t}.
\]

The integral \( \int_1^\infty \) converges for all \( s \in \mathbb{C} \), as \( \omega(t) \) is exponentially decaying as \( t \to +\infty \), so defines an entire function. We transform the integral \( \int_0^1 \) by changing variables \( t \to 1/t \) and using the functional
THE FUNCTIONAL EQUATION OF $\zeta(s)$

The functional equation of $\omega(t)$:

$$\int_0^1 \omega(t)t^{s/2} \frac{dt}{t} = \int_1^\infty \omega\left(\frac{1}{t}\right)t^{-s/2} \frac{dt}{t}$$

$$= \int_1^\infty \omega(t)t^{(1-s)/2} \frac{dt}{t} + \int_1^\infty \frac{t^{1/2} - 1}{2} t^{-s/2} \frac{dt}{t}$$

$$= \int_1^\infty \omega(t)t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.$$ 

Thus

$$\zeta^*(s) = \int_1^\infty \omega(t) \left(t^{s/2} + t^{(1-s)/2}\right) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.$$ 

This is visibly invariant under $s \mapsto 1-s$, and since $\omega(t)$ is exponentially decaying as $t \to \infty$, the integral is absolutely convergent for all $s$, and defines an analytic function there. Thus we see that $\zeta^*(s)$ is analytic for all $s$ except for simple poles at $s = 0, 1$, where

$$\text{Res}_{s=1} \zeta^*(s) = 1, \quad \text{Res}_{s=0} \zeta^*(s) = -1.$$ 

2.3. Zeros of $\zeta(s)$. We know that $\Gamma(s)$ has simple poles at the non-positive integers, while $\zeta^*(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right)\zeta(s)$ is analytic if $s \neq 0, 1$. This forces $\zeta(s)$ to have simple zeros at the negative even integers: $\zeta(-2n) = 0, n \geq 1$. These are called the “trivial zeros” of $\zeta$. Since $\Gamma(s) \neq 0$, we see that all other zeros of $\zeta$ coincide with those of $\zeta^*(s)$. We saw using the Euler product that $\zeta(s) \neq 0$ for $\text{Re}(s) > 1$, and we also saw non-vanishing on the line $\text{Re}(s) = 1$. Thus $\zeta^*(s) \neq 0$ for $\text{Re}(s) \geq 1$, and by the functional equation also $\zeta^*(s) \neq 0$ for $\text{Re}(s) \leq 0$. Hence all nontrivial zeros of $\zeta$ lie inside the critical strip $0 < \text{Re}(s) < 1$. By the functional equation, if $\zeta^*(\rho) = 0$ then also $1 - \rho$ is a zero, and since $\bar{\zeta}(s) = \zeta(\bar{s})$ we also have that $\bar{\rho}$ and $1 - \bar{\rho}$ are zeros. In particular, if a zero $\rho = 1/2 + i\gamma$ lies on the “critical line” $\text{Re}(s) = 1/2$, then $1 - \rho = 1/2 - i\gamma$ is also a zero.

The Riemann Hypothesis is that all nontrivial zeros lie on the critical line.

3. DIRICHLET L-FUNCTIONS

We now describe the functional equation of Dirichlet L-function. We start with a “primitive” Dirichlet character mod $q$. For $q$ prime, this means $\chi$ non-trivial. The functional equation relates $L(s, \chi)$ and $L(1-s, \overline{\chi})$. 

Recall the Gauss sum
\[ \tau(\chi) = \sum_{m=1}^{q} \chi(m)e^{2\pi im/q} \]
which satisfies \(|\tau(\chi)| = \sqrt{q}\) (we saw this for prime \(q\) and nontrivial \(\chi\), the same holds for arbitrary \(q\) and \(\chi\) primitive). We define \(a = a(\chi) \in \{0, 1\}\) by
\[ \chi(-1) = (-1)^a(\chi) \]
so that \(a = 0\) for even characters, and \(a = 1\) for odd characters. Define the “root number”
\[ W(\chi) := i^a \frac{\sqrt{q}}{\tau(\chi)} \]
which has absolute value one. Set
\[ L^*(s, \chi) := \left(\frac{\pi}{q}\right)^{-s/2} \Gamma \left(\frac{s + a}{2}\right) L(s, \chi). \]

**Theorem 3.1.** For a primitive character mod \(q\), \(L^*(s, \chi)\) is analytic in the entire complex plane, and satisfies
\[ L^*(1 - s, \chi) = W(\chi)L^*(s, \chi) \]

As in the case of \(\zeta\), the Dirichlet L-functions have trivial zeros at either the odd negative integers (\(\chi\) odd) or at the even non-positive integers (\(\chi\) even), and the nontrivial zeros, namely those of \(L^*(s, \chi)\). The generalized Riemann Hypothesis (GRH) states that all the nontrivial zeros lie on the critical line \(\text{Re}(s) = 1/2\).