1. Summation by parts (Abel summation)

We will often use the following technique to convert weighted partial sums of a series from knowledge of the partial sums.

**Proposition 1.1.** Let \( \{a_n\} \) be a sequence of complex numbers, and \( f(t) \in C^1(1, \infty) \) a continuously differentiable function. Let

\[
A(t) = \sum_{1 \leq n \leq t} a_n
\]

(we define \( A(t)0 \) if \( t < 1 \)). Then for integers \( 0 \leq z < x \),

\[
\sum_{z < n \leq x} a_n f(n) = A(x) f(x) - A(z) f(z) - \int_z^x A(t) f'(t) dt
\]

*Proof.* We have

\[
a_n = A(n) - A(n - 1)
\]

and so

\[
\sum_{z < n \leq x} a_n f(n) = \sum_{z < n \leq x} (A(n) - A(n - 1)) f(n)
\]

\[
= \sum_{z < n \leq x} A(n) f(n) - \sum_{z < n \leq x} A(n-1) f(n).
\]

For the second sum, change variable to write

\[
\sum_{z < n \leq x} A(n-1) f(n) = \sum_{z-1 < n \leq x-1} A(n) f(n + 1)
\]

while for the first sum, add and subtract boundary terms so that the range of summation matches that above:

\[
\sum_{z < n \leq x} A(n) f(n) = A(x) f(x) + \sum_{z-1 < n \leq x-1} A(n) f(n) - A(z) f(z).
\]

---

*Date: March 1, 2022.*
Thus we find
\[ \sum_{z < n \leq x} a_n f(n) = A(x) f(x) - A(z) f(z) - \sum_{z-1 < n \leq x-1} A(n) \{ f(n+1) - f(n) \}. \]

Using the fundamental theorem of calculus gives
\[ f(n+1) - f(n) = \int_n^{n+1} f'(t) dt. \]

Since \( A(t) = A(n) \) for \( n \leq t < n+1 \) we have
\[ A(n) \{ f(n+1) - f(n) \} = \int_n^{n+1} A(t) f'(t) dt. \]

Thus
\[ \sum_{z-1 < n \leq x-1} A(n) \{ f(n+1) - f(n) \} = \sum_{z \leq n \leq x-1} \int_n^{n+1} A(t) f'(t) dt \]
\[ = \int_z^x A(t) f'(t) dt. \]

which proves the claim. \( \square \)

**Example 1.2.** Show that
\[ \log N! = N \log N - N + O(\log N) \]

Indeed, take \( f(t) = \log t, \ a_n = 1 \), then \( A(t) = \lfloor t \rfloor \) and
\[ \log N! = \sum_{1 < n \leq N} \lfloor n \rfloor \log n = [N] \log N - [1] \log 1 - \int_1^N \frac{\lfloor t \rfloor}{t} dt \]
\[ = N \log N + O(\log N) - \int_1^N \frac{t - \{t\}}{t} dt \]
\[ = N \log N - N + O(\log N). \]

**Example 1.3.** Show
\[ \sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right) \]

for the constant \( \gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt = 0.577216\ldots \) (Euler’s constant).
Indeed, take \( f(t) = 1/t, \) \( a_n = 1 \) so that \( A(t) = \lfloor t \rfloor, \) \( z = 0, x = N \) to obtain

\[
\sum_{n=1}^{N} \frac{1}{n} = 1 + \sum_{1 < n \leq N} \frac{1}{n} = 1 + \frac{\lfloor N \rfloor}{N} - \frac{1}{1} + \int_{1}^{N} \frac{\lfloor t \rfloor}{t^2} \, dt
\]

\[
= 1 + \int_{1}^{N} \left( \frac{1}{t} - \frac{\{t\}}{t^2} \right) \, dt
\]

\[
= \log N + 1 - \int_{1}^{\infty} \frac{\{t\}}{t^2} \, dt + \int_{N}^{\infty} \frac{\{t\}}{t^2} \, dt
\]

\[
= \log N + 1 - \int_{1}^{\infty} \frac{\{t\}}{t^2} \, dt + O\left( \frac{1}{N} \right).
\]

2. Mertens’ theorems (1874)

Last week, we proved

**Theorem 2.1.**

\[
\sum_{\text{prime } p \leq x} \frac{\log p}{p} = \log x + O(1).
\]

We now deduce

**Theorem 2.2.** For a suitable constant \( C, \)

\[
\sum_{\text{prime } p \leq x} \frac{1}{p} = \log \log x + C + O\left( \frac{1}{\log x} \right).
\]

**Proof.** Take \( f(t) = 1/\log t, \) and \( a_n = \log n/n \) if \( n = p \) is prime, and \( a_n = 0 \) otherwise. Then \( f'(t) = -1/(t \log^2 t) \) and by Mertens’ first theorem, \( A(t) = \log t + O(1). \) Set

\[
R(t) = A(t) - \log t = O(1)
\]
Applying summation by parts gives
\[
\sum_{\text{prime} \leq x} \frac{1}{p} = \frac{1}{2} + \sum_{2 < n \leq x} a_n \frac{1}{\log n}
\]

\[
= \frac{1}{2} + \frac{\log x + O(1)}{\log x} - \frac{\log 2}{\log 2} + \int_2^x \frac{\log t + R(t)}{t(\log t)^2} dt
\]

\[
= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{R(t) dt}{t(\log t)^2}
\]

\[
= \log \log x + 1 - \log \log 2 + \int_2^\infty \frac{R(t) dt}{t(\log t)^2} - \int_x^\infty \frac{R(t) dt}{t(\log t)^2} + O\left(\frac{1}{\log x}\right)
\]

where
\[
C = 1 - \log \log 2 + \int_2^\infty \frac{R(t) dt}{t(\log t)^2}
\]

after noting that
\[
\left| \int_x^\infty \frac{R(t) dt}{t(\log t)^2} \right| \ll \int_x^\infty \frac{dt}{t(\log t)^2} = \frac{1}{\log x}.
\]

We arrive at another result of Mertens:

**Theorem 2.3.** For a suitable constant \( c_2 > 0 \), we have
\[
\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{c_2}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).
\]

**Proof.** Taking logarithms we have
\[
\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \sum_{p \leq x} \log(1 - \frac{1}{p}) = -\sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \left(\frac{1}{p} - \log \frac{1}{1 - \frac{1}{p}}\right).
\]

Using Theorem 2.2
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + C + O\left(\frac{1}{\log x}\right).
\]

while from the Taylor expansion
\[
\log \frac{1}{1 - x} = x + \frac{x}{2} + \cdots = x + O(x^2)
\]
we obtain
\[ \frac{1}{p} - \log \frac{1}{1 - \frac{1}{p}} = O\left(\frac{1}{p^2}\right) \]
and therefore we have a convergent sum:
\[
\sum_{p \leq x} \left( \frac{1}{p} - \log \frac{1}{1 - \frac{1}{p}} \right) = \sum_{p} \left( \frac{1}{p} - \log \frac{1}{1 - \frac{1}{p}} \right) - \sum_{p > x} \left( \frac{1}{p} - \log \frac{1}{1 - \frac{1}{p}} \right)
\]
\[= c' + O\left( \sum_{p > x} \frac{1}{p^2} \right) = c' + O\left( \frac{1}{x} \right). \]

Thus we find
\[
\log \prod_{p \leq x} (1 - \frac{1}{p}) = -\log \log x - C + O\left( \frac{1}{\log x} \right) + c' + O\left( \frac{1}{x} \right)
\]
\[= -\log \log x + c_1 + O\left( \frac{1}{\log x} \right). \]

Exponentiating we find
\[
\prod_{p \leq x} (1 - \frac{1}{p}) = \exp\left( -\log \log x + c_1 + O\left( \frac{1}{\log x} \right) \right)
\]
\[= \frac{e^{c_1}}{\log x} \exp\left( O\left( \frac{1}{\log x} \right) \right) = \frac{c_2}{\log x} \left( 1 + O\left( \frac{1}{\log x} \right) \right). \]

Remark: Mertens determined the constant to be \( c_2 = e^{-\gamma} \) where \( \gamma \) is Euler’s constant.

3. Mean values of the number of divisors and prime divisors

We set \( d(n) \) to be the number of (positive) divisors of an integer \( n \), and \( \omega(n) \) the number of distinct prime divisors. This \( d(1) = 1, \omega(1) = 0, d(2) = 2, \omega(2) = 2, d(4) = 3, \omega(4) = 2 \) etc.

The results we have obtained so far allow us to determine the mean values of these arithmetic functions.

3.1. Mean value of \( \omega(n) \).

Theorem 3.1.
\[
\frac{1}{N} \sum_{n \leq N} \omega(n) = \log \log N + O(1).
\]
This means that “on average”, an integer of size $n \approx X$ has about $\log \log X$ distinct prime factors. Hardy and Ramanujan (1917) showed that the “typical” integer has $\log \log n$ prime factors, that is that the mean represents the typical value: For all $\epsilon > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \# \{n \leq N : |\omega(n) - \log \log n| > (\log \log n)^{1/2+\epsilon} \} = 0$$

(this is often stated as “the normal order of $\omega(n)$ is $\log \log n$”).

Moreover, Erdős and Kac (1940) proved a central limit theorem, showing that for any $a < b$,

$$\lim_{N \to \infty} \frac{1}{N} \# \{n \leq N : a \leq \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq b \} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

**Exercise 1.** Let $\Omega(n)$ be the number of all prime divisors of $n$, counted with multiplicities. Thus $\Omega(4) = 2$ while $\omega(4) = 1$. Find the asymptotic mean value of $\Omega(n)$.

### 3.2. Mean value of the divisor function.

We now turn to the divisor function $d(n)$.

**Proposition 3.2.**

$$\frac{1}{N} \sum_{n=1}^{N} d(n) = \log N + O(1).$$

**Proof.** Write $d(n) = \sum_{d|n} 1$ and so when summing $d(n)$, we can exchange order of summation

$$\sum_{n=1}^{N} d(n) = \sum_{n=1}^{N} \sum_{d|n} 1 = \sum_{d \leq N} \sum_{n \leq N \atop d|n} 1$$

$$= \sum_{d \leq N} \left\lfloor \frac{N}{d} \right\rfloor.$$

We now approximate $\left\lfloor \frac{N}{d} \right\rfloor = \frac{N}{d} + O(1)$ (this approximation is good when $d$ is small, but poor when $d$ is large, and later we will make use of this observation) and obtain

$$= \sum_{d \leq N} \frac{N}{d} + O(\sum_{d \leq N} 1) = N \sum_{d=1}^{N} \frac{1}{d} + O(N).$$
Using our result
\[ \sum_{d=1}^{N} \frac{1}{d} = \log N + \gamma + O\left(\frac{1}{N}\right) \]
we obtain
\[ \sum_{n=1}^{N} d(n) = N \log N + O(N). \]

\[ \square \]

We can with some more thought obtain a finer result (this is Dirichlet’s “hyperbola method”):

**Theorem 3.3.**

\[ \frac{1}{N} \sum_{n=1}^{N} d(n) = \log N + (2\gamma - 1) + O\left(\frac{1}{\sqrt{N}}\right) \]

where \( \gamma \) is Euler’s constant.

**Proof.** We write
\[ d(n) = \sum_{x,y \geq 1 \atop x \cdot y = n} 1 \]
so that
\[ \sum_{n \leq N} d(n) = \sum_{n \leq N} \sum_{x,y \geq 1 \atop x \cdot y = n} 1 = \sum_{1 \leq x \cdot y \leq N} 1 \]
where the sum is over integer points under the hyperbola \( xy = N \).

![Figure 1. The hyperbola method](Image)
We break up the sum into subsums over the regions given in Figure 1. By symmetry, \( II = III \) so that
\[
\sum_{n \leq N} d(n) = I + II + III = I + 2 \cdot II
\]

The sum \( I \) is simple as it is the number of lattice points in a square \([1, \sqrt{N}]^2\), so equals
\[
I = ([\sqrt{N}])^2 = (\sqrt{N} + O(1))^2 = N + O(\sqrt{N}).
\]

The sum \( II \) is over points under the hyperbola \( xy \leq N \), but with \( x \) small \((x \leq \sqrt{N})\):
\[
II = \sum_{x \leq \sqrt{N}} \sum_{\sqrt{N} < y \leq N/x} 1 = \sum_{x \leq \sqrt{N}} \left( \frac{N}{x} - \sqrt{N} + O(1) \right)
\]
\[
= N \sum_{x \leq \sqrt{N}} \frac{1}{x} - (\sqrt{N} + O(1))^2
\]

Using the result of Example 1.3 on the partial sum of the harmonic series, we obtain
\[
II = N \left( \log \sqrt{N} + \gamma + O\left( \frac{1}{\sqrt{N}} \right) \right) - N + O(\sqrt{N})
\]
and substituting in (3.1) gives the result. \( \square \)

It is worth noting that, unlike for the function \( \omega(n) \), for the divisor function, the mean value does not represent the typical behaviour! The normal order of \( \log d(n) \) is \( \log 2 \cdot \log \log n \), so that for almost all integers, the size of \( d(n) \) is about \((\log n)^{\log 2 + o(1)}\), while the average is \( \log n \).

**Remark.** Understanding the true size of the difference
\[
\Delta(x) := \sum_{n \leq x} d(n) - x (\log x + (2\gamma - 1))
\]
is known as Dirichlet’s divisor problem, and is a notoriously difficult issue. We showed that \( |\Delta(x)| \ll x^{1/2} \). The exponent \( 1/2 \) has been improved several times, beginning with \( 1/3 \) by Voronoi (1904) with many subsequent improvements. The conjecture is that \( |\Delta(x)| \ll x^{1/4+o(1)} \)

3.3. **The maxima of \( \omega(n) \) and of \( d(n) \).** For the divisor function, we clearly have \( d(n) = 2 \) infinitely often (for prime \( n \)). For an upper bound, we have the very useful result:

**Theorem 3.4.**
\[
d(n) \ll n^\varepsilon, \quad \forall \varepsilon > 0.
\]
Moreover, for all $a > 1$,
\[ d(n) \ll 2^{a \log 2 \log n / \log \log n} = n^{B / \log \log n}. \]

**Proof.** To see this, we need to know that $d(n)$ is multiplicative. Given that, we write $n = \prod_j p_j^{k_j}$, and take $\epsilon = \epsilon(n) > 0$. Then by multiplicativity, since $d(p^k) = k + 1$,
\[ \frac{d(n)}{n^\epsilon} = \prod_j \frac{k_j + 1}{p_j^{k_j \epsilon}}. \]

Now if $p_j \geq 2^{1/\epsilon}$ then the factors are $\leq 1$, because then $p_j^{k_j \epsilon} \geq 2$ and
\[ \frac{k_j + 1}{p_j^{k_j \epsilon}} \leq \frac{k_j + 1}{2^{k_j}} \leq 1 \]
because $2^{k} \geq k + 1$ for $k \geq 1$.

Assume first that $\epsilon$ is fixed, independent of $n$. Then we are left with the product of the bounded primes $p_j < 2^{1/\epsilon}$, the number of which is, say, $J(\epsilon)$ and
\[ \frac{d(n)}{n^\epsilon} \leq \prod_{p_j < 2^{1/\epsilon}} \frac{k_j + 1}{p_j^{k_j \epsilon}} \leq \prod_{j=1}^{J(\epsilon)} \frac{k_j + 1}{2^{k_j \epsilon}} =: C(\epsilon) \]
(using $p_j \geq 2$) as claimed.

To improve this, we take
\[ \epsilon(n) = \frac{A \log 2}{\log \log n}. \]

We want to bound the product $\prod_{p_j < 2^{1/\epsilon}} \frac{k_j + 1}{p_j^{k_j \epsilon}}$. We have
\[ \frac{k_j + 1}{p_j^{k_j \epsilon}} \leq 1 + \frac{k_j}{p_j^{k_j \epsilon}} \]
and $p^{\epsilon} > \log p^{\epsilon} \geq \log 2^{k \epsilon} = k \epsilon \log 2$ so that $k/p^{\epsilon} < 1/(\epsilon \log 2)$, so that
\[ \frac{k_j + 1}{p_j^{k_j \epsilon}} < 1 + \frac{1}{\epsilon \log 2} \]
and therefore
\[ \prod_{p_j < 2^{1/\epsilon}} \frac{k_j + 1}{p_j^{k_j \epsilon}} < \prod_{p_j < 2^{1/\epsilon}} \left( 1 + \frac{1}{\epsilon \log 2} \right). \]
Using $1 + x \leq e^x$ for $x > 0$ gives
\[
\prod_{p_j < 2^{1/\epsilon}} e^{1/(\epsilon \log 2)} = e^{\pi(2^{1/\epsilon})/\epsilon \log 2} < \exp\left(\frac{2^{1/\epsilon}}{\epsilon \log 2}\right).
\]

Since
\[
2^{1/\epsilon} = \exp(\log 2 \cdot \frac{\log \log n}{A \log 2}) = (\log n)^{1/A}
\]
we find
\[
d(n) < n^\epsilon \exp\left(\frac{\log \log n (\log n)^{1/A}}{A (\log 2)^2}\right) < \exp(\epsilon \log n + O((\log n)^{1-\delta})) < \exp(c \frac{\log n}{\log \log n})
\]

\[\square\]

**Theorem 3.5.**
\[
\omega(n) \ll \frac{\log n}{\log \log n}.
\]

**Proof.** Observe that
\[
\omega(n) \leq \log_2 d(n)
\]
since $d(n) = \prod_j (k_j + 1) \geq \prod_j 2 = 2^{\omega(n)}$. Therefore, using the first bound $d(n) \ll n^\epsilon$ of Theorem 3.4 we obtain $\omega(n) < \epsilon \log n$ for all $\epsilon > 0$, that is $\omega(n) = o(\log n)$, or using $d(n) \ll \exp(a \log 2 \log n/ \log \log n)$ gives
\[
\omega(n) \leq \log_2 d(n) \ll \frac{\log n}{\log \log n}.
\]

\[\square\]