NUMBER THEORY SEMINAR 2014/15
EXERCISE 4
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DUE DATE: JANUARY 15 , 2015

1. A probability measure μ on the unit circle S^1 is symmetric if it is invariant under complex conjugation $z \mapsto \bar{z}$ and under rotation by 90 degrees $z \mapsto \sqrt{-1}z$. Show that the Fourier coefficients $\widehat{\mu}(k)$ of a symmetric measure must vanish unless $4 \mid k$. (The Fourier coefficients of a probability measure on S^1 are defined as $\widehat{\mu}(k) := \int_0^{2\pi} e^{ik\theta} d\mu(\theta)$).

2. For an integer n which is a sum of two squares, let

$$r_2(n) = \#\{\lambda \in \mathbb{Z}^2 : |\lambda|^2 = n\}$$

be the number of representations of n as a sum of two squares, and we define a symmetric probability measure on S^1 by

$$\mu_n = \frac{1}{r_2(n)} \sum_{\substack{\lambda \in \mathbb{Z}^2 \\ |\lambda|^2 = n}} \delta_{\lambda/\sqrt{n}}$$

where δ_z is the delta-measure at a point $z \in S^1$.

Show that if m, n are coprime and are sums of two squares, then μ_{mn} is the convolution of the measures μ_m and μ_n:

$$\mu_{mn} = \mu_m * \mu_n \ .$$

3. (This problem might be harder, so can be handed in at the end of the semester):

For $Y \gg 1$, we define a set of integers

$$\mathcal{M}(Y) := \{Y < m \leq 2Y : 4m^2 + 1 \text{ is prime}\} \ .$$

It can be shown by an upper bound sieve that

$$\#\mathcal{M}(Y) \ll \frac{Y}{\log Y} \ .$$

It is conjectured that $\#\mathcal{M}(Y) \to \infty$ as $Y \to \infty$. We shall assume this conjecture.

Define

$$n(Y) := \prod_{m \in \mathcal{M}(Y)} (4m^2 + 1) \ .$$

a) If $\lambda \in \mathbb{Z}^2$, $|\lambda|^2 = n(Y)$, show that there is some $k = k(\lambda) \in \{0, 1, 2, 3\}$ so that

$$\Big| \frac{\lambda}{\sqrt{n(Y)}} - (\sqrt{-1})^k \Big| \ll \frac{1}{\log Y}$$

b) Show that the measures $\mu_{n(Y)}$ weakly converge to an average of 4 delta functions at $\pm 1, \pm\sqrt{-1}$ as $Y \to \infty$:

$$\mu_{n(Y)} \Rightarrow \frac{1}{4}\Big(\delta_1 + \delta_{-1} + \delta_{\sqrt{-1}} + \delta_{-\sqrt{-1}}\Big) .$$

That is, for any continuous function f on S^1,

$$\lim_{Y\to\infty} \frac{1}{r_2(n(Y))} \sum_{\substack{\lambda\in\mathbb{Z}^2 \\ |\lambda|^2=n(Y)}} f(\frac{\lambda}{\sqrt{n(Y)}}) = \frac{1}{4}\Big(f(1)+f(-1)+f(\sqrt{-1})+f(-\sqrt{-1})\Big) .$$