EXERCISE 2
NUMBER THEORY RESEARCH SEMINAR 2017/18
PROF. ZEÉV RUDNICK
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1. Let $\pi(x) = \#\{p \le x : p \text{ prime}\}$, and

$$\psi(x) := \sum_{n \le x} \Lambda(n) = \sum_{\substack{k \ge 1,\, p \text{ prime} \\ p^k \le x}} \log p$$

where

$$\Lambda(n) = \begin{cases} \log p, & n = p^k,\ p \text{ prime}, k \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

is the von Mangoldt function. Show that $\pi(x) \sim x/\log x$ as $x \to \infty$ if and only if $\psi(x) \sim x$.

To do this, look up summation by parts (Abel summation).

2. The "Approximate Explicit Formula" says that

$$\psi(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ \rho=\beta+i\gamma \\ |\gamma|<T}} \frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^2}{T}\Big)$$

where the sum is over the nontrivial zeros of the Riemann zeta function.

The Riemann Hypothesis (RH) says that all nontrivial zeros lie on the critical line $\mathrm{Re}\rho = 1/2$. Show that RH implies that

$$\psi(x) = x + O\Big(x^{1/2+\varepsilon}\Big), \qquad \forall \varepsilon > 0$$

You will need to use the Riemann-von Mangoldt formula

$$N(T) := \{\rho = \beta + i\gamma : 0 < \gamma \le T\} \sim \frac{T}{2\pi} \log T, \qquad T \to \infty$$

3. Show that assuming the Riemann Hypothesis, for any $\varepsilon > 0$, if $x > x_0$ is sufficiently large, then there is a prime p between x and $x + y$ if $y \gg x^{1/2+\varepsilon}$.