EXERCISE 3
NUMBER THEORY RESEARCH SEMINAR 2017/18
DUE DATE: JANUARY 4, 2018

1. In class we explained why the least common multiple of the first N integers is given by

$$\log \operatorname{lcm}\Big(\{1, 2, \dots, N\}\Big) = \psi(N) \sim N, \quad N \to \infty$$

Let $f(x) = x(x+1) \in \mathbb{Z}[x]$. Show that

$$\log \operatorname{lcm}\Big(\{f(n) : n = 1, 2, \dots, N\}\Big) \sim N, \quad N \to \infty$$

2. For the polynomial ring $\mathbb{F}_q[T]$, show that the degree of the least common of all monic polynomials of degree n is given by

$$\deg \operatorname{lcm}\{f \in \mathbb{F}_q[T] : \deg f = n,\ f \text{ monic}\} = \sum_{j=1}^{n} q^j$$

3. For a polynomal $f \in \mathbb{Z}[x]$, let

$$\rho_f(m) := \#\{t \bmod m : f(t) = 0 \bmod m\}$$

be the number of solutions of the the congruence $f(x) = 0 \bmod m$. For the polynomial $f(x) = x^2 + 1$, show that

$$\rho_f(p) = \begin{cases} 2, & p = 1 \bmod 4 \\ 0, & p = 3 \bmod 4 \\ 1, & p = 2 \end{cases}$$

Then show that if $p \neq 2$ is an odd prime, then $\rho_f(p^k) = \rho_f(p)$ for any $k \geq 1$.