0.1. Arithmetic functions and Dirichlet series

An arithmetic function is simply a function $a : \mathbb{N} \to \mathbb{C}$ on the natural numbers. Examples are the constant function $1$, the power functions $n^\alpha$.

The Dirichlet convolution of two arithmetic functions $a, b : \mathbb{N} \to \mathbb{C}$ is defined by

$$a * b(n) := \sum_{d \mid n} a(d)b(\frac{n}{d})$$

For instance, if we denote by $1$ the constant function $1$, and

$$\sigma_0(n) = \#\{d \geq 1, d \mid n\} = \sum_{d\mid n} 1$$

the number of divisors of $n$, then clearly

$$1 * 1 = \sigma_0$$

Likewise, the divisor sums

$$\sigma_\alpha(n) = \sum_{d\mid n} d^\alpha$$
are clearly the convolution

$$\sigma_\alpha = 1 * n^\alpha$$

To any arithmetic function $a : \mathbb{N} \to \mathbb{C}$, which is of polynomial growth: $|a(n)| \ll n^A$, we associate a Dirichlet series

$$D_{a}(s) := \sum_{n \geq 1} a(n) n^{-s}, \quad \text{Re}(s) > A + 1$$

which converges for $\text{Re}(s) > A$, and uniformly in any closed half-plane $\text{Re}(s) \geq A + \delta$, hence defines a holomorphic function for $\text{Re}(s) > A$.

For instance, for the constant function $1$ we have

$$D_{1}(s) = \sum_{n \geq 1} \frac{1}{n^{s}} = \zeta(s)$$

is the Riemann zeta function.

For the power function $n^\alpha$, we have

$$\sum_{n \geq 1} \frac{n^\alpha}{n^{s}} = \zeta(s - \alpha), \quad \text{Re}(s) > 1 + \text{Re}(\alpha)$$

**Lemma 0.1.** The Dirichlet series attached to a convolution is the product of the two Dirichlet series:

$$D_{a*b}(s) = D_{a}(s)D_{b}(s)$$

**Corollary 0.2.** The Dirichlet series $D_{a}(s) = \sum_{n \geq 1} \sigma_{\alpha}(n)n^{-s}$ associated to the divisors sum $\sigma_{\alpha}$ is the product

$$D_{\alpha}(s) = \zeta(s)\zeta(s - \alpha)$$

An arithmetic function is multiplicative if $a(1) \neq 0$ and $a(mn) = a(m)a(n)$ whenever $m, n$ are coprime. Necessarily then $a(1) = 1$.

Examples: the constant function $1$, the power functions $n^\alpha$ are multiplicative (in fact completely multiplicative).

**Lemma 0.3.** If $a, b : \mathbb{N} \to \mathbb{C}$ are multiplicative then so is their convolution $a * b$.

Since $\sigma_{\alpha} = 1 * n^\alpha$, we obtain

**Lemma 0.4.** The divisor sums $\sigma_{\alpha}$ are multiplicative: $\sigma_{\alpha}(mn) = \sigma_{\alpha}(m)\sigma_{\alpha}(n)$ if $\gcd(m, n) = 1$.

**Exercise 1.** The divisor sums $\sigma_{\alpha}$ satisfy for $p$ prime and $r \geq 1$

$$\sigma_{\alpha}(p)\sigma_{\alpha}(p^r) = \sigma_{\alpha}(p^{r+1}) + p^{\alpha}\sigma_{\alpha}(p^{r-1})$$

**Lemma 0.5.** If $a : \mathbb{N} \to \mathbb{C}$ is multiplicative then for $\text{Re}(s) \gg 1$,

$$D_{a}(s) = \prod_{p \text{ prime}} \sum_{r \geq 0} a(p^r)p^{-rs}$$
0.2. The Dirichlet series attached to a modular form. Let \( f \neq 0 \in M_k \) be a modular form of weight \( k \), with Fourier expansion \( f = \sum_{n \geq 0} a_f(n)q^n \). The associated Dirichlet series is, for \( s \in \mathbb{C} \), \( \text{Re}(s) \gg 1 \)

\[
D(s, f) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}
\]

Note that we ignore the zero’th coefficient \( a_f(0) \). Recall that we showed that for any \( F \in M_k \), the Fourier coefficients satisfy \( |a_f(n)| \ll n^{k-1} \) (and better bounds for cusp forms). Hence the series converges for \( \text{Re}(s) > k \), and uniformly in any closed half-plane \( \text{Re}(s) \geq k + \delta \), hence defines a holomorphic function for \( \text{Re}(s) \gg 1 \).

0.3. Eisenstein series. The normalized Eisenstein series has Fourier expansion

\[
E_k = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n
\]

Hence the associated Dirichlet series is (a multiple of)

\[
D(s) = \sum_{n \geq 1} \sigma_{k-1}(n)q^n
\]

We have already computed this Dirichlet series, so that we find

\[
D(s, E_k) = \zeta(s)\zeta(s - (k - 1))
\]

0.4. Analytic continuation and functional equation. The Gamma function is defined for \( \text{Re}(s) > 0 \) as

\[
\Gamma(s) := \int_0^{\infty} e^{-t}t^{s-1}dt
\]

Integration by parts shows that in this regime, we have the functional equation

\[
\Gamma(s + 1) = s\Gamma(s)
\]

so that in particular we obtain that for \( n \geq 1 \) integer, \( \Gamma(n + 1) = n! \) and using the functional equation we obtain that \( \Gamma \) has meromorphic continuation to the entire complex plane, save for simple poles at the non-negative integers \( s = 0, -1, -2, -3, \ldots \).

Exercise 2. Compute the residue \( \text{Res}_{s=-n} \Gamma(s) \).

Theorem 0.6. Let \( f \in S_k \) be a cusp form \( (k \geq 12 \text{ even}) \). Set \( \Lambda_f(s) := (2\pi)^{-s}\Gamma(s)D(s, f) \), initially defined for \( \text{Re}(s) \gg 1 \). Then \( D(s, f) \) admits an analytic continuation to the entire complex plane, and satisfies the functional equation

\[
\Lambda_f(s) = i^{-k}\Lambda_f(k - s)
\]
Proof. We first give an integral representation of $\Lambda_f(s)$. Consider the integral (Mellin transform)

$$I(s) = \int_0^{\infty} f(iy)y^s\frac{dy}{y}$$

Recall that a cusp form decays exponentially at infinity: $|f(x + iy)| \ll e^{-2\pi y}$ as $y \to +\infty$, so the integral converges at $y = \infty$ for all $s$, and for $y \to 0$, we use the modular transformation formula $f(-1/\tau) = \tau^k f(\tau)$ for $\tau = iy$

$$f(iy) = \left(\frac{1}{iy}\right)^k f\left(-\frac{1}{iy}\right)$$

to deduce that $|f(iy)| \ll y^{-k}e^{-2\pi/y}$ as $y \to 0$, so that the integral also converges at $y = 0$ if for all $s$. Hence $I(s)$ is an entire function.

Now insert the Fourier expansion

$$f(iy) = \sum_{n \geq 1} a_f(n)e^{-2\pi ny}$$

to find

$$I(s) = \sum_{n \geq 1} a_f(n) \int_0^{\infty} e^{-2\pi ny}y^s\frac{dy}{y}$$

Changing variables gives

$$\int_0^{\infty} e^{-2\pi ny}y^s\frac{dy}{y} = (2\pi n)^{-s}\Gamma(s)$$

so that

$$I(s) = (2\pi)^{-s}\Gamma(s) \sum_{n \geq 1} a_f(n)n^{-s} =: \Lambda_f(s)$$

which shows that $\Lambda_f(s)$ is entire.

It remains to prove the functional equation. Separate the integral as

$$I(s) = \int_1^{y=0} + \int_1^{\infty}$$

Using the transformation formula for $f$, we write

$$\int_0^{1} f(iy)y^s\frac{dy}{y} = \int_0^{1} \left(\frac{1}{iy}\right)^k f\left(-\frac{1}{iy}\right)y^s\frac{dy}{y} = i^{-k} \int_0^{1} f\left(\frac{i}{y}\right)y^{s-k}\frac{dy}{y}$$

Now change variables $y' = 1/y$:

$$= i^{-k} \int_1^{\infty} f(iy')(y')^{k-s}\frac{dy'}{y'}$$

so that we obtain

$$I(s) = \int_1^{\infty} f(iy)\left(y^s + i^{-k}y^{k-s}\right)\frac{dy}{y}$$
Hence, using $i^{-k} = i^k$ for $k$ even,
\[
I(k - s) = \int_1^\infty f(iy) \left( y^{k-s} + i^{-k} y^s \right) \frac{dy}{y} = i^k I(s)
\]

\[\square\]

0.5. Analytic continuation and functional equation for Rie-
mann’s zeta function. The above proof of the functional equation
for Dirichlet series attached to cusp forms is modeled on one of Rie-
mann’s proofs of the corresponding fact for the Riemann zeta function,
except that there is an extra step which leads to a pole. The result is

**Theorem 0.7.** Let $\zeta^*(s) := \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s)$. Then $\zeta^*(s)$ is analytic except for simple poles at $s = 0$ and $s = 1$, and has a functional equation
\[
\zeta^*(s) = \zeta^*(1 - s)
\]

**Proof.** The completed Riemann zeta function $\zeta^*$ is essentially the Mellin transform of the one variable theta function
\[
\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau}
\]
Precisely, set
\[
\omega(y) = \frac{1}{2} \left( \frac{\theta(iy) - 1}{\sqrt{y}} \right) = \sum_{n \geq 1} e^{-\pi n^2 y}
\]
and consider the integral
\[
I(s) := \int_0^\infty \omega(y) y^{s/2} \frac{dy}{y}
\]
The integral converges for all $s$ at $y = \infty$, since $\omega(y) \ll e^{-\pi y}$ as $y \to \infty$.
To understand convergence at $y = 0$, recall the transformation for-
ma of the one variable theta function
\[
\theta(-1/\tau) = \sqrt{-i\tau} \theta(\tau)
\]
In particular, taking $\tau = iy$ (which is the way we proved it...)
\[
\theta \left( \frac{i}{y} \right) = \sqrt{\pi} \theta(iy)
\]
so that as $y \to 0$, since $\theta(iy) \to 1$ as $y \to 0$,
\[
\theta(iy) \sim \frac{1}{\sqrt{y}}
\]
and hence
\[
\omega \left( \frac{1}{y} \right) \sim \frac{1}{2\sqrt{y}}
\]
Thus the integral $I(s)$ converges at $y = 0$ like that of $\int_0^1 \frac{1}{2\sqrt{y}} y^{s/2} \frac{dy}{y}$, namely for $\text{Re}(s) > 1$.

Then for $\text{Re}(s) > 1$,

$$I(s) = \sum_{n \geq 1} \int_0^\infty e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} = \sum_{n \geq 1} (\pi n^2)^{-s/2} \Gamma\left(\frac{s}{2}\right) =: \zeta^*(s)$$

Now split the integral as

$$I(s) = \int_0^1 + \int_1^\infty$$

The integral $\int_1^\infty$ is absolutely convergent for all $s \in \mathbb{C}$ so is an entire function. To treat the integral $\int_0^1$, change variables $y = 1/t$

$$\int_0^1 \omega(y) y^{s/2} \frac{dy}{y} = \int_1^\infty \omega\left(\frac{1}{t}\right) t^{-s/2} \frac{dt}{t}$$

The functional equation of theta $\theta(i/t) = \sqrt{t} \theta(it)$ gives

$$\omega\left(\frac{1}{t}\right) = \frac{1}{2} \left( \sqrt{t} \theta(it) - 1 \right) = \sqrt{t} \omega(t) + \frac{\sqrt{t} - 1}{2}$$

Hence

$$\int_1^\infty \omega\left(\frac{1}{t}\right) t^{-s/2} \frac{dt}{t} = \int_1^\infty \omega(t) t^{1-s} \frac{dt}{t} + \int_1^\infty \frac{\sqrt{t} - 1}{2} t^{-s/2} \frac{dt}{t}$$

The first integral converges for all $s$, hence is an entire function of $s$, while the second integral is explicitly evaluated as

$$\int_1^\infty \frac{\sqrt{t} - 1}{2} t^{-s/2} \frac{dt}{t} = -\frac{1}{1-s} - \frac{1}{s}$$

Thus

$$\zeta^*(s) = -\frac{1}{1-s} - \frac{1}{s} + \int_1^\infty \omega(t) \left( t^{\frac{s}{2}} + \frac{1-s}{2} t^{\frac{1-s}{2}} \right) \frac{dt}{t}$$

Both summands are clearly symmetric under $s \mapsto 1-s$, so that $\zeta^*(s) = \zeta^*(1-s)$, and the integral is entire, and so we find that $\zeta^*(s)$ has an analytic continuation to all of $\mathbb{C}$ except for simple poles at $s = 0, 1$. □

**Corollary 0.8.** The Riemann zeta function has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$, where $\text{Res}_{s=1} \zeta = 1$.

This is because $\Gamma \neq 0$ and so $\zeta$ does not have any more poles than $\zeta^*$.

**Corollary 0.9.** $\zeta(-2n) = 0$ for $n = 1, 2, \ldots$. 
These are called the “trivial” zeros of ζ(s). The nontrivial zeros are the zeros of ζ∗.

0.6. Euler products for Hecke eigenforms. Assume that \( f \in S_k \) is a normalized Hecke eigenform: \( T(n)f = \lambda_f(n)f, \ a_f(1) = 1 \), so that the Fourier expansion is

\[
f(\tau) = \sum_{n \geq 1} \lambda_f(n)q^n
\]

The corresponding Dirichlet series is then

\[
D(s, f) = \sum_{n \geq 1} \lambda_f(n)n^{-s}
\]

Since \( \lambda_f \) is multiplicative, we have

\[
D(s, f) = \prod_{\text{prime } p} \sum_{r \geq 0} \lambda_f(p^r)p^{-rs}
\]

Lemma 0.10.

\[
\sum_{r=0}^{\infty} \lambda_f(p^r)X^r = \frac{1}{1 - \lambda_f(p)X + p^{k-1}X^2}
\]

This is equivalent to the recursion

\[
\lambda_f(p)\lambda_f(p^r) = \lambda_f(p^{r+1}) + p^{k-1}\lambda_f(p^{r-1}), \quad r \geq 1
\]

Corollary 0.11. Let \( f \in S_k \) be a cuspidal Hecke eigenform. Then

\[
D(s, f) = \prod_p (1 - \lambda_f(p)p^{-s} + p^{k-1}p^{-2s})^{-1}
\]

0.7. The Riemann Hypothesis for \( L(s, f) \). If \( f \in S_k \) is a Hecke eigenform, then we saw that the corresponding Dirichlet series admits an Euler product

\[
D(s, f) = \sum_{n \geq 1} \lambda_f(n)n^{-s} = \prod_p (1 - \lambda_f(p)p^{-s} + p^{k-1-2s})^{-1}
\]

Writing the \( p \)-the factor as

\[
1 - \lambda_f(p)X + p^{k-1}X^2 = (1 - \alpha_1(p)p^{(k-1)/2}X)((1 - \alpha_2(p)p^{(k-1)/2}X)
\]

\[
= \det(I - Xp^{(k-1)/2}A_p)
\]

where

\[
A_p := \begin{pmatrix} \alpha_1(p) & 0 \\ 0 & \alpha_2(p) \end{pmatrix}, \quad p^{k-1} \text{tr } A_p = \lambda_f(p), \quad \det A_p = 1
\]
we have

\[ D(s, f) = \prod_p \det(I - p^{-s} p^{(k-1)/2} A_p)^{-1} \]

Deligne’s theorem (Ramanujan’s conjecture) \(|\lambda_f(p)| \leq 2p^{(k-1)/2}\) is equivalent to \(A_{p} \in \text{SU}(2)\) is unitary.

Lets normalize differently: set

\[ L(s, f) := D\left(s + \frac{k-1}{2}, f\right) = \prod_p \det(I - p^{-s} A_p)^{-1} \]

and

\[ L^*(s, f) = (2\pi)^{-\left(s+\frac{k-2}{2}\right)} \Gamma(s + \frac{k-1}{2}) D\left(s + \frac{k-1}{2}, f\right) \]

which now satisfies a functional equation

\[ L^*(s, f) = i^k L^*(1 - s, f) \]

whose symmetry axis is the line \(\text{Re}(s) = \frac{1}{2}\). The analogue of the Riemann Hypothesis is that all zeros of \(L^*(s, f)\) (which are called the non-trivial zeros of \(L(s, f)\)) lie on the line \(\text{Re}(s) = 1/2\).

This has not been established in any example.

0.8. The converse theorem. We saw that a modular form gives a Dirichlet series with analytic continuation and a certain specific functional equation. It turns out that Dirichlet series with this precise functional equation must correspond to modular forms. This is Hecke’s “converse theorem” (1936) for \(\text{SL}(2, \mathbb{Z})\).

**Theorem 0.12.** Let \(D(s) = \sum_{n \geq 1} a(n)n^{-s}\) be a Dirichlet series, with \(|a(n)| \ll n^\nu\) for some \(\nu > 0\) (so is absolutely convergent in \(\text{Re}(s) \gg 1\)), so that

1. \(D(s)\) admits an analytic continuation to all of \(\mathbb{C}\)
2. \(D(s)\) satisfies the functional equation \((k \geq 12\) even\)
   \[ \Lambda(s) := (2\pi)^{-s}\Gamma(s) D(s) = (-1)^{k/2} D(k - s), \]
3. \(D(s)\) is bounded in vertical strips: Given \(-\infty < \alpha < \beta < +\infty\), there is some \(C(\alpha, \beta)\) so that \(|D(\sigma + it)| \leq C(\alpha, \beta)\) for \(\sigma \in [\alpha, \beta]\).

Then there is some \(f \in S_k\) so that \(D(s) = D(s, f)\).

The proof is a simple application of Mellin inversion. One forms the function \(f(\tau) := \sum_{n \geq 1} a(n)q^n\), which by definition satisfies \(f(\tau + 1) = f(\tau)\) and is holomorphic in \(|q| < 1\) (i.e. \(\tau \in \mathbb{H}\)) since \(|a(n)| \ll n^\nu\) and vanishes at \(q = 0\), so all that is left is to establish the transformation
formula \( f(-1/\tau) = \tau^k f(\tau) \). Since both sides are analytic in \( \tau \), it suffices to do so for \( \tau = iy, y > 0 \). This is done by using Mellin inversion

\[
e^{-t} = \frac{1}{2\pi i} \int_{\text{Re}(s)=2} \Gamma(s) t^{-s} ds
\]

so that

\[
f(iy) = \sum_{n \geq 1} a(n) e^{-2\pi ny} = \sum_{n \geq 1} a(n) \frac{1}{2\pi i} \int_{\text{Re}(s)=2} \Gamma(s)(2\pi ny)^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{\text{Re}(s)=2} \Gamma(s)(2\pi)^{-s} D(s) y^{-s} ds = \frac{1}{2\pi i} \int_{\text{Re}(s)=2} \Lambda(s) y^{-s} ds
\]

Now use the functional equation \( \Lambda(s) = i^k \Lambda(k - s) \), change variables, shift contours and eventually recover \( (iy)^{-k} f(i/y) \). Along the way one needs to use that \( D(s) \) is bounded in vertical strips.