MODULAR FORMS 2019: FOURIER COEFFICIENTS I WEEK OF APRIL 14 2019

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Contents

1. The Fourier expansion of E_k and Δ

One of the most important aspects in applications of the theory of modular forms is their "Fourier coefficients", that is the coefficients in the q-expansion

$$f(\tau) = \tilde{f}(q) = \sum_{n\geq 0} a(n)q^n$$

Exercise 1. Let $f, g \in M_k$, so that the first $\lfloor \frac{k}{12} \rfloor + 1$ Fourier coefficients coincide (that is $f = \sum_{n\geq 0} a(n)q^n$, $g = \sum_{n\geq 0} b(n)q^n$ and $a(n) = b(n)$ for all $0 \leq n \leq \lfloor \frac{k}{12} \rfloor$). Then $f = g$.

1.1. The q-expansion of G_k.

We determine the Fourier expansion of the Eisenstein series G_k. Define the divisor sums

$$\sigma_s(n) := \sum_{d|n} d^s$$

Theorem 1.1. *For $k > 2$ even,*

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

We present two different proofs

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Proof. We write

$$G_k(\tau) = \sum_{n=0}^{\infty} A_n e^{2\pi i n\tau}$$

We already saw that $A_0 = 2\zeta(k)$. For $n \geq 1$, write

$$A_n = \int_0^1 G_k(x+iy)e^{-2\pi i n(x+iy)}dx = \sum_{(c,d)\neq(0,0)} \int_0^1 (c\tau+d)^{-k}e^{-2\pi i n\tau}dx$$

The terms with $c = 0$ give

$$\sum_{d\neq 0} d^{-k}\int_0^1 e^{-2\pi i n(x+iy)}dx = 2\zeta(k)\delta_{n,0} = 0$$

since we assume $n \neq 0$.

For the terms with $c \neq 0$, recall that k is even to replace the sum over $c \neq 0$ by twice the sum over $c \geq 1$

$$\sum_{c\neq 0} = 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\int_0^1 (c\tau+d)^{-k}e^{-2\pi i n\tau}dx$$

For fixed $c \geq 1$, write $d = cq + r$ with $q \in \mathbb{Z}$ and $1 \leq r \leq c$, so that

$$\begin{aligned}
\sum_{d\in\mathbb{Z}}\int_0^1 (c\tau+d)^{-k}e^{-2\pi i n\tau}dx &= \sum_{q\in\mathbb{Z}}\sum_{r \bmod c}\int_0^1 (c(\tau+q)+r)^{-k}e^{-2\pi i n(\tau+q)}dx \\
&= \sum_{r \bmod c}\int_{-\infty}^{\infty}(c\tau+r)^{-k}e^{-2\pi i n\tau}dx \\
&= c^{-k}\sum_{r \bmod c}\int_{-\infty}^{\infty}(\tau+\frac{r}{c})^{-k}e^{-2\pi i n\tau}dx \\
&= c^{-k}\sum_{r \bmod c}e^{2\pi i n\frac{r}{c}}\int_{-\infty}^{\infty}\tau^{-k}e^{-2\pi i n\tau}dx \\
&= \begin{cases} (\frac{n}{c})^{k-1}t_k, & c \mid n \\ 0, & c \nmid n \end{cases}
\end{aligned}$$

where $t_k = \int_{-\infty}^{\infty}\tau^{-k}e^{-2\pi i\tau}dx$.

Now summing over $c \geq 1$ gives for $n \geq 1$

$$A_n = 2t_k\sum_{c|n}(\frac{n}{c})^{k-1} = 2t_k\sum_{d|n}d^{k-1} = 2t_k\sigma_{k-1}(n)$$

Finally, we need to evaluate the integral t_k, which can be shown to equal

$$t_k := \int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i\tau} dx = (2\pi)^{k-1} \int_{-\infty}^{\infty} \tau^{-k} e^{-i\tau} dx = \frac{(-2\pi i)^k}{(k-1)!}$$

which concludes the proof. □

Here is a different proof:

Proof. We write

$$G_k(\tau) = 2\zeta(k) + 2\sum_{m=1}^{\infty} R_m(\tau), \quad R_m(\tau) := \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

We will need the partial fraction expansion of the cotangent function (see your Complex Variables course):

$$\pi \cot \pi\tau = \frac{1}{\tau} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right)$$

On the other hand, writing $q = e^{2\pi i\tau}$, we have

$$\pi \cot(\pi\tau) = \pi i \frac{e^{i\pi\tau} + e^{-i\pi\tau}}{e^{i\pi\tau} - e^{-i\pi\tau}} = \pi i - \frac{2\pi i}{1-q} = -\pi i - 2\pi i \sum_{d=1}^{\infty} q^d$$

Hence we find

$$\frac{1}{\tau} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right) = -\pi i - 2\pi i \sum_{d=1}^{\infty} q^d$$

Differentiate this $k-1$ times (w.r.t. τ, recall $q = e^{2\pi i\tau}$) and divide by $(-1)^{k-1}(k-1)!$ to obtain

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i d\tau}$$

Replace τ by $m\tau$ for $m \geq 1$ to obtain

$$R_m(\tau) := \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i m d\tau}$$

Hence

$$G_k(\tau) = 2\zeta(k) + 2\sum_{m=1}^{\infty} R_m(\tau) = 2\zeta(k) + 2\frac{(-2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i m d\tau}$$

Finally, in the double sum we write $n = dm$ and change order of summation to obtain

$$\sum_{d=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}e^{2\pi imd\tau} = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d^{k-1}\Big)q^n = \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

which gives the claim. □

We define

$$E_k(\tau) = \frac{1}{2}\sum_{\gcd(m,n)=1}(m\tau+n)^{-k}$$

which satisfies $E_k(i\infty) = 1$, so that $G_k = 2\zeta(k)E_k$. Then

$$E_k(\tau) = 1 + \gamma_k\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where (see Table 1).

$$\gamma_k = \Big(\frac{(2\pi i)^k}{(k-1)!}\Big)/\zeta(k)$$

k	4	6	8	10	12	14	16
γ_k	240	-504	480	-264	65520/691	-24	16320/3617

TABLE 1. The numbers γ_k.

Euler showed that for $k \geq 2$ even

$$\zeta(k) = \frac{2^{k-1}}{k!}B_{\frac{k}{2}}\cdot\pi^k$$

where B_m are the Bernoulli numbers (Table 2)

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{m=1}^{\infty}(-1)^{m+1}B_m\frac{x^{2m}}{(2m)!}$$

m	1	2	3	4	5	6	7
B_m	1/6	1/30	1/42	1/30	5/66	691/2730	7/6

TABLE 2. The Bernoulli numbers

Therefore

$$\gamma_k = \frac{(2\pi i)^k/(k-1)!}{2^{k-1}B_{k/2}\pi^k/k!} = (-1)^{k/2}\frac{2k}{B_{k/2}}$$

and in particular

$$\gamma_4 = (-1)^2 \frac{2 \cdot 4}{1/30} = 240, \quad \gamma_6 = (-1)^3 \frac{2 \cdot 6}{1/42} = -504$$

so that

Corollary 1.2.

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

1.2. **The q-expansion of Δ.**

Theorem 1.3.

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^n$$

with $\tau(1) = 1$, and $\tau(n) \in \mathbb{Z}$ are integers.

Proof. We have $\Delta = g_2^3 - 27 g_3^2$ and we saw that $g_2(i\infty) = 4\pi^4/3$, $g_3(i\infty) = 8\pi^6/27$, so that $g_2 = \frac{4\pi^4}{3} E_4$, $g_3 = (\frac{2\pi}{3})^3 E_6$. Therefore

$$\Delta = (\frac{4\pi^4}{3} E_4)^3 - 27(\frac{2\pi}{3})^6 E_6^2 = \frac{2^6 \pi^{12}}{27} \left(E_4^3 - E_6^2 \right)$$

Inserting the q-expansions of E_4 and E_6 (Corollary 1.2) gives

$$\begin{aligned} E_4^3 - E_6^2 &= \left(1 + 240q + O(q^2\right)^3 - \left(1 - 504q + O(q^2)\right)^2 \\ &= \left(1 + 3 \cdot 240q + O(q^2)\right) - \left(1 - 2 \cdot 504q + O(q^2)\right) \\ &= 1728q + O(q^2) \end{aligned}$$

Hence (note $1728 = 12^3$)

$$\Delta = \frac{2^6 \pi^{12}}{27} \left(12^3 q + O(q^2) \right) = (2\pi)^{12} q + O(q^2)$$

which shows that $\tau(1) = 1$.

More generally, define $\tau(n)$ by

$$E_4^3 - E_6^2 = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^3 - \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right)^2 =: 1728 \sum_{n=1}^{\infty} \tau(n) q^n$$

then

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^n$$

We write

$$\sum_{n=1}^{\infty} \tau(n) q^n = \frac{E_4^3 - E_6^2}{1728} = \frac{(1+240A)^3 - (1-504B)^2}{1728}$$

where

$$A := \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad B := \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

have q-expansions with integer coefficients. Expanding and simplifying gives

$$\sum_{n=1}^{\infty} \tau(n) q^n = 5\frac{A-B}{12} + B + 100A^2 + 8000A^3 - 147B^2$$

and we are reduced to checking that the q-series $(A-B)/12$ has integer coefficients, which we leave as an exercise. □

Exercise 2. Show that

$$\sigma_3(n) = \sigma_5(n) \bmod 24$$

The first few values of $\tau(n)$ are given in Table 3

n	1	2	3	4	5	6	7	8
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744	84480
n	9	10	11	12	13	14	15	16
$\tau(n)$	-113643	-115920	534612	-370944	-577738	401856	1217160	987136

TABLE 3. The Ramanujan τ function.

1.3. **Properties of the Ramanujan tau-function $\tau(n)$.** Jacobi's product formula assets that

$$\boxed{\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}}$$

Ramanujan (1916) observed, but did not prove, the following three properties of $\tau(n)$:

(1) $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m,n) = 1$
(2) $\tau(p)\tau(p^r) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1})$ for p prime and $r \geq 1$.
(3) $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

The first two properties were proved by Mordell (1917), and can be combined as

$$\tau(m)\tau(n) = \sum_{d|\gcd(m,n)} d^{11}\tau(\frac{mn}{d^2})$$

The third one, called the Ramanujan conjecture, was proved by Deligne in 1973 as a consequence of his proof of the Weil conjectures.

The Sato-Tate conjecture (Mikio Sato 1963, based on computer experiments): For p prime $\tau(p)/\sqrt{p^{11}} =: 2\cos\theta_p$ is distributed like the trace of a random SU(2) matrix. That is

$$\lim_{x\to\infty} \frac{1}{\#\{p \le x\}} \#\{p \le x : \alpha < \theta_p < \beta\} = \frac{2}{\pi}\int_\alpha^\beta (\sin\theta)^2 d\theta$$

This was proved by Barnet-Lamb, Geraghty, Harris, Taylor (2009).

Lehmer's conjecture (1947): $\tau(n) \neq 0$ for all n. This is still open.

1.4. **The q-expansion of j.** Using the information about the Fourier coefficients of E_4 and Δ, we may compute the coefficients of the q-expansion of the j-invariant

$$j = 1728\frac{g_2^3}{\Delta} = \frac{E_4^3}{\Delta/(2\pi)^{12}}$$

Since $E_4 = 1+240\sum_{n\ge1}\sigma_3(n)q^n$ and $\Delta/(2\pi)^{12} = q\Big(1+\sum_{n\ge2}\tau(n)q^{n-1}\Big)$ have integer coefficients, we find

$$j = \frac{E_4^3}{\Delta/(2\pi)^{12}} = \frac{1}{q}\frac{1+240\sum_{n\ge1}\sigma_3(n)q^n}{1+\sum_{n\ge2}\tau(n)q^{n-1}} = \frac{1}{q} + \sum_{n\ge0} c(n)q^n$$

and therefore $c(n) \in \mathbb{Z}$ are integers.

The first few terms in the expansion are

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots$$

The Fourier coefficients $c(n)$ grow much faster than those of modular forms. Asymptotically,

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$$

(Petersson 1932).

In 1978, John McKay observed that the coefficient $c(1)$ satisfies

$$196884 = 1 + 196883$$

with the significance that the number 196883 is the dimension of the smallest irreducible representation of the monster group, the largest sporadic simple group, having order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \cdot 10^{53}$$

This led to the theory of "monstrous moonshine", asserting that the $c(n)$'s are the dimensions of the graded part of an infinite-dimensional graded algebra representation of the monster group called the moonshine module.