MODULAR FORMS 2019: PETERSSON'S FORMULA WEEKS OF APRIL 29 AND MAY 5 2019

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Contents

1. Hecke's bound for Fourier coefficients of cusp forms

A key quantity associated to a modular form are its Fourier coefficients, that is the coefficients $a_f(n)$ of its q-expansion

$$f(\tau) = \sum_{n\geq 0} a_f(n)q^n.$$

For instance, we computed the coefficients of the Eisenstein series $G_k(\tau)$, which for $n \neq 0$ are divisor sums $c_k\sigma_{k-1}(n)$. In particular, they are at least of size n^{k-1}. In fact, this is also an upper bound, since for $k-1>1$ we have

$$\sigma_{k-1}(n) \ll n^{k-1}$$

Exercise 1. Let $\sigma_s(n) = \sum_{d|n} d^s$ (the sum over all divisors of n).
a) Show that σ_s is multiplicative: $\sigma_s(mn) = \sigma_s(m)\sigma_s(n)$ if m, n are coprime
b) Show that for $s > 1$, $\sigma_s(n) \ll_s n^s$.

As we shall see, the coefficients of *cusp forms* are much smaller, and that is a key input into the theory of modular forms.

Before beginning this study, we need some preparations.

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Lemma 1.1. *Let $f \in S_k$ be a cusp form. Then*
a) $f(x+iy) \ll_f e^{-2\pi y}$ decays exponentially $y \to +\infty$.
b) There is some $C_f > 0$ so that $y^k|f(x+iy)|^2 \le C_f$ for all $x+iy \in \mathbb{H}$.

Proof. a) Using the q-expansion $f(\tau) = \sum_{n\ge 1} a_f(n) q^n$, which has no constant term when f is cuspidal, we see that $f = O(q)$ as $q \to 0$, which gives $f(x+iy) \ll_f e^{-2\pi y}$ as $y = \operatorname{Im}\tau \to +\infty$.

In particular, there is some $C_f > 0$ so that $y^k|f(x+iy)|^2 \le C_f$ for $y \ge \sqrt{3}/2$.

b) We note that the quantity $h(\tau) = y^k|f(x+iy)|^2$ is invariant under $\mathrm{SL}(2,\mathbb{Z})$: $h(\gamma\tau) = h(\tau)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, because $y = \operatorname{Im}(\tau)$ transforms as

$$\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus $h(\tau)$ is determined by its restriction to the standard fundamental domain $\mathcal{F}$, and

$$\sup_{\tau\in\mathbb{H}} h(\tau) = \sup_{\tau\in\mathcal{F}} h(\tau)$$

Since $h \le C_f$ for $\operatorname{Im}(\tau) \ge \sqrt{3}/2$, in particular $h \le C_f$ for all τ in the standard fundamental domain, and by periodicity of h it therefore satisfies this bound for all $\tau \in \mathbb{H}$. $\square$

We can now present a bound for Fourier coefficients of cusp forms, due to Erich Hecke (1930's)

Theorem 1.2. *(Hecke) Let $f \in S_k$ be a cusp form of weight k, with q-expansion $f = \sum_{n\ge 1} a_f(n) q^n$. Then*

$$|a_f(n)| \le c_f n^{k/2}$$

Compare to the lower bound of n^{k-1} for the coefficients $\sigma_{k-1}(n)$ of the Eisenstein series, which are much bigger for large n.

Proof. Write the Fourier series of f as

$$f(x+iy) = \sum_{m\ge 1} a_f(m) e^{-2\pi m y} e^{2\pi i m x}$$

Hence by Parseval (orthogonality of the exponentials $e^{2\pi i m x}$), for all $y > 0$

$$\int_0^1 y^k |f(x+iy)|^2 dx = y^k \sum_{m=1}^{\infty} |a_f(m)|^2 e^{-4\pi m y}$$

By Lemma 1.1, we know $y^k |f(x+iy)|^2 \leq C_f$ so that we obtain

$$y^k \sum_{m=1}^{\infty} |a_f(m)|^2 e^{-4\pi m y} \leq C_f$$

for all $y > 0$.

In particular, if we truncate the sum after n terms (all terms are non-negative so we can do this) for any $n \geq 1$ we have an inequality

$$\sum_{m=1}^{n} |a_f(m)|^2 e^{-4\pi m y} \leq C_f y^{-k}$$

Now choose $y = 1/n$, and replace $e^{-4\pi m/n} \geq e^{-4\pi}$ for $m \leq n$ to obtain

$$\boxed{\sum_{m=1}^{n} |a_f(m)|^2 \ll n^k} \tag{1}$$

and in particular $|a_f(n)| \ll_f n^{k/2}$ as claimed. $\square$

The inequality (1) indicates that the expected size of $a_f(n)$ is $n^{(k-1)/2}$, rather than $n^{k/2}$. This was dubbed the Ramanujan conjecture, and eventually proven by Deligne in 1973: $|a_f(n)| \ll n^{(k-1)/2+\varepsilon}$, for all $\varepsilon > 0$. We will return to this later.

Corollary 1.3. *Let $f \in M_k$ be a modular form of weight $k > 2$, not necessarily cuspidal. Then the Fourier coefficients satisfy*

$$|a_f(n)| \ll n^{k-1}$$

Indeed, writing $f = AG_k + g$ with $g \in S_k$ cuspidal, we see that

$$a_f(n) = c\sigma_{k-1}(n) + a_g(n)$$

Applying Hecke's bound for $a_g(n) \ll n^{k/2}$ and the bound $\sigma_{k-1}(n) \ll n^{k-1}$ for $k-1 > 1$ gives the result.

2. The Petersson inner product

The hyperbolic measure on the upper half plane is the one with volume element $dxdy/y^2$. A computation shows that it is invariant under any hyperbolic isometry (Mobius transformation) $g \in \mathrm{SL}(2,\mathbb{R})$.

Exercise 2. Check that $dxdy/y^2$ is invariant under an Mobius transformation $g \in \mathrm{SL}(2,\mathbb{R})$.

Exercise 3. Compute the hyperbolic area $\int_{\mathcal{F}} \frac{dxdy}{y^2}$ of the standard fundamental domain $\mathcal{F}$ for $\mathrm{SL}(2,\mathbb{Z})$.

The Petersson inner product is defined on S_k by (write $\tau = x+iy$)

$$\langle f,g\rangle := \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}y^k\frac{dxdy}{y^2}$$

Check that the insertion of the factor y^k makes this well defined, that is independent of which fundamental domain we integrate over, because $f(\tau)\overline{g(\tau)}y^k$ is $\mathrm{SL}(2,\mathbb{Z})$-invariant, and the measure $dxdy/y^2$ is invariant under any hyperbolic isometry (Mobius transformation) $g \in \mathrm{SL}(2,\mathbb{R})$.

The integral is convergent whenever one of the forms is cuspidal (and the other is a modular form), since cusp forms decay exponentially $|f(x+iy)| \ll e^{-2\pi y}$ as $y \to +\infty$ as is seen from the q-expansion $f(\tau) = a_f(1)q + O(q^2)$, while any modular form is bounded as $y \to +\infty$.

We will need to use the following "unfolding" principle:

Lemma 2.1. *If h is any "nice" function on $\mathbb{H}$, which is invariant under Γ_∞ (that is periodic: $h(\tau+1) = h(\tau)$) then*

$$\int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(\gamma\tau)\frac{dxdy}{y^2} = \int_0^\infty \Big(\int_0^1 h(x+iy)dx\Big)\frac{dy}{y^2} \tag{2}$$

Proof. If $\mathcal{F}$ is a fundamental domain for Γ, and $\{\gamma_j\}$ are represnstatives for the disjoint cosets of Γ_∞ in Γ¡ then $\cap_j\gamma_j\mathcal{F}$ is a fundamental domain for Γ_∞. Hence

$$\int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(\gamma\tau)\frac{dxdy}{y^2} = \int_{\Gamma_\infty\backslash\mathbb{H}} h(\tau)\frac{dxdy}{y^2}$$

where now the integration is over a fundamental domain for Γ_∞.

As a fundamental domain for $\Gamma_\infty = \{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} : n \in \mathbb{Z}\}$, which is just the subgroup of translations generated by $\tau \mapsto \tau+1$, we can take the strip

$$\{x+iy : 0 \le x < 1, y > 0\}$$

Hence

$$\int_{\Gamma_\infty\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(\gamma\tau)\frac{dxdy}{y^2} = \int_0^\infty \Big(\int_0^1 h(x+iy)dx\Big)\frac{dy}{y^2}$$

as claimed. $\square$

Using the Petersson inner product, we can see that the Poincaré series are the dual of the Fourier coefficients

Theorem 2.2. *Let $f \in S_k$ be a cusp form, with Fourier expansion*

$$f(\tau) = \sum_{n=1}^\infty a_f(n)q^n.$$

Then for $m > 0$,

$$\langle f, P_m^k\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m)$$

Proof. We write

$$\begin{aligned}
\langle f, P_m^k\rangle &= \int_{\Gamma\backslash\mathbb{H}} f(\tau) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{\overline{e(m\gamma(\tau))}}{\overline{(c\tau+d)}^k} \operatorname{Im}(\tau)^k \frac{dxdy}{y^2} \\
\text{using } f(\tau) &= f(\gamma(\tau))/(c\tau+d)^k \text{ gives} \\
&= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma(\tau))\frac{\operatorname{Im}(\tau)^k}{|c\tau+d|^{2k}} \overline{e(m\gamma(\tau))}\frac{dxdy}{y^2} \\
\text{using } \operatorname{Im}(\gamma(\tau)) &= \operatorname{Im}(\tau)/|c\tau+d|^2 \text{ gives} \\
&= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma(\tau))\overline{e(m\gamma(\tau))} \operatorname{Im}(\gamma(\tau))^k \frac{dxdy}{y^2}
\end{aligned}$$

Applying the unfolding principle (2) with $h(\tau) = f(\tau)\overline{e(m\tau)} \operatorname{Im}(\tau)^k$ (which is clearly periodic) gives

$$\begin{aligned}
\langle f, P_m^k\rangle &= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma(\tau))\overline{e(m\gamma(\tau))} \operatorname{Im}(\gamma(\tau))^k \frac{dxdy}{y^2} \\
&= \int_0^\infty \Big(\int_0^1 f(x+iy)e(m(-x+iy))y^k dx\Big)\frac{dy}{y^2} \\
&\quad = \int_0^\infty \Big(\int_0^1 f(x+iy)e(-mx)dx\Big)y^{k-1}e^{-2\pi my}\frac{dy}{y}
\end{aligned}$$

Inserting the Fourier expansion $f(x+iy) = \sum_{n\geq 1} a_f(n)e^{-2\pi ny}e(nx)$ gives

$$\langle f, P_m^k \rangle = a_f(m)\int_0^\infty e^{-4\pi my}y^{k-1}\frac{dy}{y} = a_f(m)\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

$\square$

We deduce that for $k \geq 12$, the $P_m^k \in S_k$ span S_k, because a modular form is determined by it's Fourier coefficients (the q-expansion) and by Theorem 2.2 a cusp for which is orthogonal to all the Poincaré series must be zero.

Corollary 2.3. *The space of cusp forms admits a basis consisting of Poincaré series.*

Exercise 4. Let $k \geq 12$. Then the Poincaré series $\{P_m^k, 1 \leq m \leq 1 + \frac{k}{12}\}$ span S_k.

3. Petersson's formula

(without proof)

Hans Petersson (1932) gave a formula for the Fourier coefficients of the Poincaré series P_m^k as an infinite sum of "Kloosterman sums" weighted by Bessel functions. This awkward-looking formula is in fact very useful, allows to bring in consideration from the theory of exponential sums and the "Riemann Hypothesis for curves over a finite field", and we shall use it to improve Hecke's bound for the Fourier coefficients of any cusp form.

Theorem 3.1. *Let $m \geq 1$, and $k > 2$ even. The q-expansion of the Poincaré series is $P_m^k(\tau) = \sum_{n\geq 1} p_m(n)q^n$ where*

$$p_m(n) = 2\pi i^k (\frac{n}{m})^{\frac{k-1}{2}} \sum_{c\geq 1} \frac{\mathrm{Kl}(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$$

Here $J_{k-1}(x)$ is the Bessel function of order $k-1$, and $\mathrm{Kl}(m,n;c)$ is a Kloosterman sum. We discuss both below.

The proof of Petersson's formula, which we omit, runs along the lines of our computation of the Fourier coefficients of the Eisenstein series, but is more complicated, using a parameterization of the double cosets $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$. Recall that in our computation we encountered the integral $\int_{-\infty}^{\infty} \tau^{-k} e^{-i\tau} d\tau = 2\pi i^k/(k-1)!$ (this is easy); here we will come up against the integral

$$\int_{-\infty}^{\infty} z^{-k} e^{-i\lambda(z+z^{-1})} dz = 2\pi i^{-k} J_{k-1}(2\lambda).$$

3.0.1. *The J-Bessel function.* We also recall the J-Bessel function $J_\alpha(x)$ (for us α will be a positive integer): These are solutions of Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

that are finite at the origin ($x = 0$) for integer or positive α. They admit a power series expansion around $x - 0$

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \Big(\frac{x}{2}\Big)^{2m+\alpha}$$

In particular, near $x = 0$ we have

$$J_\alpha(x) \sim \frac{1}{\alpha!}\Big(\frac{x}{2}\Big)^\alpha \ll x^\alpha, \quad 0 < x \ll 1$$

An integral representation used by Bessel, is that for $\alpha = n$ integer

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(x\sin t - nt)}dt$$

For $x \to +\infty$, we have

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}}\Big(\cos(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + O(\frac{1}{x})\Big) \ll x^{-1/2}$$

The Bessel function admits an integral representation that is especially relevant for us

$$J_{k-1}(4\pi t) = \frac{i^k}{2\pi}\int_{-\infty+iy}^{+\infty+iy} z^{-k}e^{-2\pi i t(z+\frac{1}{z})}dz \tag{3}$$

where $z = x + iy$ with $y > 0$.

3.0.2. *Kloosterman sums.* Let $c \geq 1$ and $m, n \in \mathbb{Z}$ integers. The Kloosterman sum is defined as

$$\mathrm{Kl}(m,n;c) := \sum_{\substack{x \bmod c \\ \gcd(x,c)=1}} e\Big(\frac{mx + n\bar{x}}{c}\Big)$$

where we abbreviate

$$e(z) := e^{2\pi i z}$$

and $\bar{x}$ denotes the multiplicative inverse of x mod c: $x\bar{x} = 1 \bmod c$. In view of the integral representation (3), they can be viewed as finite field analogues of the Bessel function.

Clearly Kl is symmetric in m and n. It satisfied a property called "twisted multiplicativity" which for some purposes allows to reduce its study to the case when c is a prime power:

Exercise 5. (Twisted Multiplicativity). Let $c = c_1c_2$ with $\gcd(c_1, c_2) = 1$. Choose n_1 and n_2 such that $n_1c_1 = 1 \bmod c_2$ and $n_2c_2 = 1 \bmod c_1$. Then

$$\mathrm{Kl}(a,b;c) = \mathrm{Kl}\left(n_2a, n_2b; c_1\right)\mathrm{Kl}\left(n_1a, n_1b; c_2\right).$$

When $m = n = 0 \bmod c$, then $\mathrm{Kl}(m,n;c) = \varphi(c)$ (the Euler totient function), and is roughly of size c. For prime modulus $c = p$, with a, b coprime to P, André Weil showed in 1948, as a consequence of his proof of the Riemann Hypothesis for curves over a finite field, that

$$|\,\mathrm{Kl}(a,b;p)| \leq 2\sqrt{p}$$

Using the general properties of Kloosterman sums, such as twisted multiplicativity, and a reduction of the case of prime power moduli to Gauss sums and the prime modulus case, one deduces

Theorem 3.2.

$$|\operatorname{Kl}(a,b;c)| \leq \sqrt{\gcd(a,b,c)}\sigma_0(c)\sqrt{c}$$

where $\sigma_0(c) = \sum_{d|c} 1$ *is the number of divisors of* c. *In particular, for fixed* $a, b \neq 0$, *we have for all* $\varepsilon > 0$ *there is some* $C(\varepsilon) > 0$ *so that*

$$|\operatorname{Kl}(a,b;c)| \leq C(\varepsilon)c^{1/2+\varepsilon} \tag{4}$$

3.1. **Kloosterman's bound.** It turns out that the hardest case to handle in bounding Kloosterman sums is that of prime modulus. We clearly have $\operatorname{Kl}(0,0;p) = p-1$.

Exercise 6. If p is prime and $a \neq 0 \bmod p$ then $\operatorname{Kl}(a,0;p) = -1$.

In the case that both $a, b \neq 0 \bmod p$, Kloosterman (1926) already gave a non-trivial bound:

Theorem 3.3. *If* $p > 2$ *is a prime and* $a, b \neq 0 \bmod p$ *then*

$$|\operatorname{Kl}(a,b;p)| \ll p^{3/4}.$$

Proof. We look at the fourth moment

$$V_4 = \sum_{a,b \bmod p} |\operatorname{Kl}(a,b;p)|^4$$

We will show that

$$V_4 \ll p^4 \tag{5}$$

On the other hand, we separate the term $(a,b) = (0,0)$ which contributes $(p-1)^4$, the $2(p-1)$ terms $\operatorname{Kl}(a,0;p)$ and $\operatorname{Kl}(0,b;p)$, which of which contributes $(-1)^4$, and the remaining ones:

$$V_4 = (p-1)^4 + 2(p-1) + \sum_{a,b\neq 0} |\operatorname{Kl}(a,b;p)|^4$$

We next use

Exercise 7. If $a, b \neq 0 \bmod p$ then

$$\operatorname{Kl}(a,b;p) = \operatorname{Kl}(ab,1;p)$$

Therefore

$$\sum_{a,b\neq 0} |\operatorname{Kl}(a,b;p)|^4 = \sum_{a,b\neq 0} |\operatorname{Kl}(ab,1;p)|^4 = (p-1)\sum_{c\neq 0} |\operatorname{Kl}(c,1;p)|^4$$

after changing variables $c = ab$. Thus we find

$$(p-1)\sum_{c\neq 0} |\operatorname{Kl}(c,1;p)|^4 = V_4 - p^4 - 2(p-1) \tag{6}$$

On the other hand, expanding out

$$|\operatorname{Kl}(a,b;p)|^4 = \sum_{x_1,x_2,y_1,y_2\neq 0} e_p(a(x_1+x_2-y_1-y_2)+b(x_1^{-1}+x_2^{-1}-y_1^{-1}-y_2^{-1}))$$

(where $e_p(x) = e^{2\pi i x/p}$), summing over $a, b \bmod p$ and switching order of summation gives

$$\begin{aligned} V_4 &= \sum_{x_1,x_2,y_1,y_2\neq 0} \sum_{a \bmod p} e_p(a(x_1+x_2-y_1-y_2) \sum_{b \bmod p} e_p(b(x_1^{-1}+x_2^{-1}-y_1^{-1}-y_2^{-1})) \\ &= p^2\#\{x_1,x_2,y_1,y_2 \neq 0 \bmod p : x_1+x_2 = y_1+y_2,\ x_1^{-1}+x_2^{-1} = y_1^{-1} = y_2^{-1}\} \end{aligned}$$

on using

$$\sum_{a \bmod p} e_p(at) = \begin{cases} p & , t = 0 \bmod p \\ 0, & t \neq 0 \bmod p \end{cases}$$

Now given y_1, y_2, if $y_2 \neq -y_1$ then x_1, x_2 are determined up to order by the system

$$x_1+x_2 = y_1+y_2, \quad x_1^{-1}+x_2^{-1} = y_1^{-1} = y_2^{-1} \tag{7}$$

as the solutions of

$$x^2 - (y_1+y_2)x + (y_1+y_2)/(y_1^{-1}+y_2^{-1}) = 0$$

which gives $2((p-1)^2-(p-1)) \leq 2p^2$ solutions, and if $y_2 = -y_1$ then necessarily $x_2 = -x_1$, which gives $(p-1)^2$ solutions. Altogether we find that the system (7) has at most $3p^2$ solutions. Hence

$$V_4 \leq 3p^4$$

Inserting (6) we obtain for $p \gg 1$ that

$$(p-1)\sum_{c\neq 0} |\operatorname{Kl}(c,1;p)|^4 \leq 2p^4$$

Dropping all terms but one (which we may as we are summing non-negative quantities) gives fr $p \gg 1$

$$|\operatorname{Kl}(a,1;p)|^4 \leq 2p^3$$

which gives $|\operatorname{Kl}(a,1;p)| \leq 2^{1/4}p^{3/4}$. □

As we have mentioned, Weil showed that $|\operatorname{Kl}(a,1;p)| \leq 2\sqrt{p}$ for p prime. Nick Katz (1988) showed that for large p, as we vary over a coprime to p, the normalized sums $-\operatorname{Kl}(a,1;p)/2\sqrt{p}$ become equidistributed with respect to the Sato-Tate measure, that is the $p-1$ numbers $\{\operatorname{Kl}(a;1;0)\sqrt{p} : 1 \leq a \leq p-1\}$ have the same distribution as the trace of a random matrix in the compact Lie group SU(2), equivalently writing $-\operatorname{Kl}(a,1;p) = 2\sqrt{p}\cos\theta(a,p)$ then as $p \to \infty$, for any subinterval $[\alpha,\beta] \subset [0,\pi]$,

$$\lim_{p\to\infty} \frac{1}{p-1}\#\{a \in [1,p-1] : \theta(a,p) \in [\alpha,\beta]\} = \frac{2}{\pi}\int_\alpha^\beta (\sin\theta)^2 d\theta$$

It is conjectured that an analogous statement holds for the angles of $\operatorname{Kl}(1,1;p)$ as p varies.

3.2. **Application to bounding Fourier coefficients of cusp forms.** We use Petersson's formula to give an upper bound for the Fourier coefficients $|p_m(n)|$. To emphasize the arithmetic input, we highlight the dependence on bounds for Kloosterman sums. Suppose that we are given a bound

$$|\operatorname{Kl}(m,n;c)| \ll c^{1-\delta} \tag{8}$$

whenever $\gcd(m,n,c) = 1$.

Proposition 3.4. *Assume the bound* (8)*. Then for fixed $m \geq 1$, and $n \to \infty$,*

$$|p_m(n)| \ll n^{\frac{k-1}{2}} \cdot n^{\frac{1-\delta}{2}}$$

Before proving Proposition 3.4, we deduce:

Corollary 3.5. *Assume the bound* (8)*. Then for any cusp form $f \in S_k$, $f = \sum_{n\geq 1} a_f(n)q^n$, we have the bound*

$$|a_f(n)| \ll_f n^{\frac{k}{2}-\frac{\delta}{2}}$$

Proof. As we saw, the space of cusp forms admits a basis of Poincaré series P_m^k, $m \in \mathcal{M} = \{m_1,\ldots,m_d\}$, $d = \dim S_k$. Then we write f as a linear combination $f = \sum_{m\in\mathcal{M}} c_m P_m^k$ to obtain

$$a_f(n) = \sum_{m\in\mathcal{M}} c_m p_m(n)$$

and inputting Proposition 3.4 gives $|a_f(n)| \ll n^{\frac{k}{2}-\frac{\delta}{2}}$. □

The trivial bound $|\operatorname{Kl}(m,n;c)| \leq c$ ($\delta = 0$) therefore recovers Hecke's bound (Theorem 1.2) $|a_f(n)| \ll n^{k/2}$.

Kloosterman's bound ($\delta = 1/4$, Theorem 3.3) gives the better estimate

$$|a_f(n)| \ll n^{\frac{k-1}{2}+\frac{3}{8}}$$

Inserting Weil's bound (essentially $\delta = 1/2$) (4) gives

$$|a_f(n)| \ll n^{\frac{k-1}{2}+\frac{1}{4}}$$

However, this falls short of the Ramanujan conjecture (proved by Deligne in 1973):

$$|a_f(n)| \ll n^{\frac{k-1}{2}}\sigma_0(n) \ll n^{\frac{k-1}{2}+\varepsilon}, \quad \forall\varepsilon > 0$$

3.2.1. *Proof of Proposition 3.4.*

Proof. For simplicity, take $m = 1$. Using Petersson's formula we obtain

$$|p_1(n)| \ll n^{(k-1)/2}\sum_{c\geq 1}\frac{|\operatorname{Kl}(1,n;c)|}{c}J_{k-1}(\frac{4\pi\sqrt{n}}{c})$$

For the Kloosterman sum, use the bound $|\operatorname{Kl}(1,n;c)| \ll c^{1-\delta}$. For the Bessel function, use $J_{k-1}(x) \ll x^{k-1}$ for $x = \sqrt{n}/c < 1$, and for $x = \sqrt{n}/c \geq 1$ use $J_{k-1}(x) \ll x^{-1/2}$:

$$\begin{aligned}|p_1(n)| &\ll n^{(k-1)/2}\sum_{1\leq c\leq\sqrt{n}} c^{-\delta}(\frac{\sqrt{n}}{c})^{-1/2} + n^{(k-1)/2}\sum_{c>\sqrt{n}} c^{-\delta}(\frac{\sqrt{n}}{c})^{k-1} \\ &= n^{\frac{k}{2}-\frac{3}{4}}\sum_{1\leq c\leq\sqrt{n}} c^{1/2-\delta} + n^{k-1}\sum_{c>\sqrt{n}}\frac{1}{c^{k-1+\delta}} \ll n^{\frac{k-1}{2}}\cdot n^{\frac{1-\delta}{2}}\end{aligned}$$

□