Poisson statistics for sequences modulo one

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אוניברסיטת חיפה

Based on joint works with C. Aistleitner (Graz), S. Baker (Birmingham) and N. Technau (Madison)

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Outline

1. Background
   - Uniform Distribution Modulo One
   - Gap Distribution
   - Correlations

2. Results
   - The Sequences (n^α) and (α^n)
   - Proof Ideas

Uniform Distribution Modulo One

- Let (x_n) be a sequence of real numbers. We are interested in the distribution of the fractional parts $\{x_n\}$ in the unit interval $[0, 1)$.
- We say that (x_n) is *uniformly distributed modulo* 1, if for every interval $I \subseteq [0, 1)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \leq n \leq N : \{x_n\} \in I\} = |I| .$$

  In other words, every interval gets its proportional share of fractional parts.
- Example: Take $x_n = \sqrt{2}n$. The fractional parts are:

$$0.414214, 0.828427, 0.242641, 0.656854, 0.0710678, 0.485281,$$
$$0.899495, 0.313708, 0.727922, 0.142136, \ldots$$

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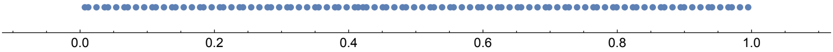

Figure: The first 100 fractional parts of the sequence $\left(\sqrt{2}n\right)$.

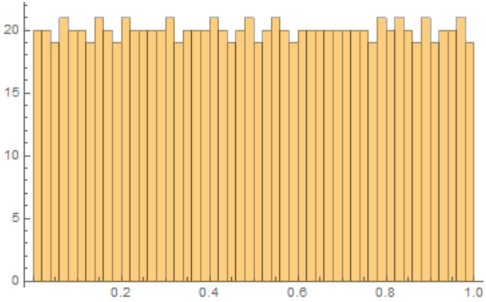

Figure: The distribution of the first 1000 fractional parts of the sequence $\left(\sqrt{2}n\right)$.

The Weyl Criterion

Theorem (The Weyl criterion, 1914)

The sequence (x_n) is uniformly distributed mod 1 if and only if for every fixed integer $k \neq 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0.$$

Theorem (Bohl, Sierpiński, Weyl, 1909-10)

Let $\alpha \in \mathbb{R}$ be an irrational number. Then the sequence (αn) is uniformly distributed mod 1.

Proof (Weyl, 1914): Fix integer $k \neq 0$. Then

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n \alpha} \right| = \frac{\left| 1 - e^{2\pi i k N \alpha} \right|}{N \left| 1 - e^{2\pi i k \alpha} \right|} \leq \frac{2}{N \left| 1 - e^{2\pi i k \alpha} \right|} = \frac{1}{N \left| \sin \pi k \alpha \right|}.$$

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Examples

Theorem (Weyl, 1914)

Let $p(x) = a_m x^m + \cdots + a_1 x + a_0$ be a polynomial with real coefficients, with at least one of the coefficients a_i $(i \geq 1)$ irrational. Then the sequence $(p(n))$ is u.d. mod 1.

- *Example: Take $p(x) = \sqrt{2}x^2$. The sequence $\left(\sqrt{2}n^2\right)$ is u.d. mod 1.*

Theorem (Fejér, Csillag 1930)

For any $\alpha \neq 0$ and $\sigma > 0$ with σ not integer, the sequence (αn^σ) is u.d. mod 1.

- *Example: The sequence $(\sqrt{n})$ is u.d. mod 1.*

Theorem (Koksma, 1935)

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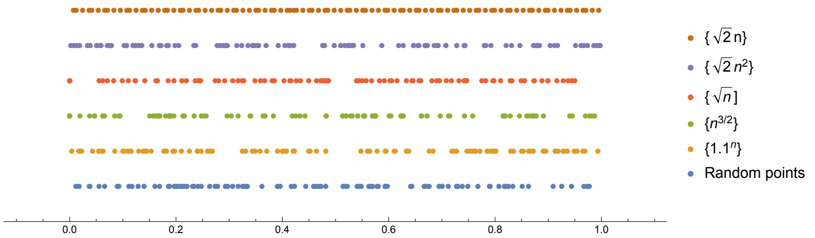

Figure: Some uniformly distributed sequences modulo one

- The definition of uniform distribution mod 1 does not capture the fact that some sequences look much more equidistributed than others. It also fails to fully capture the "randomness" of the sequences.

Fine-Scale Statistics

- Uniform distribution modulo 1 measures the number of points in *fixed* intervals $I \subseteq [0, 1)$.
- Subtler aspects of a sequence may be revealed by considering fine-scale statistics, i.e., statistics in intervals of length $1/N$ (the average gap), after restricting to the first N elements of the sequence.
- To study "randomness" of a sequence, we can compare these statistics to those of random points in the unit interval.
- Let $X_1, X_2, \ldots, X_N$ be i.i.d. random variables uniformly distributed in the unit interval. The *order statistics* $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)}$ are distributed like the first N points of a Poisson point process $\mathcal{N}(t)$ on $\mathbb{R}$ (conditional on the event $\mathcal{N}(1) = N$).

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Gap Distribution

- For the first N elements of a sequence (x_n), denote the *ordered* fractional parts by

$$x^N_{(1)} \leq x^N_{(2)} \leq \cdots \leq x^N_{(N)} \leq x^N_{(N+1)}$$

where by convention $x^N_{(N+1)} = 1 + x^N_{(1)}$.

- Let $\delta^N_n = N\left(x^N_{(n+1)} - x^N_{(n)}\right)$ be the normalized gaps.
- Poisson statistics: For any $I \subseteq [0, \infty)$

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{1 \leq n \leq N : \delta^N_n \in I\right\} = \int_I e^{-s}\, ds.$$

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The Three-Gap Theorem

- Consider the gaps in the sequence (αn) .

Theorem (Sós, Surányi, Świerczkowski, 1957–1959)

Let $\alpha \in \mathbb{R}$. For any N, consider the gaps δ_n^N of the fractional parts of the sequence (αn). Then δ_n^N attains at most three distinct values.

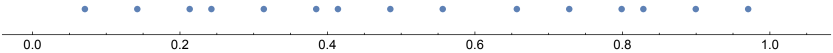

Figure: The first 15 fractional parts of the sequence $\sqrt{2}n$. The (rescaled) gaps are: 1.06602, 1.06602, 0.441559, 1.06602, 1.06602, 0.441559, 1.06602, 1.06602, 1.50758, 1.06602, 1.06602, 0.441559, 1.06602, 1.06602, 1.50758.

The Sequence $\left(\alpha n^d\right)$

- We say that an irrational α is *Diophantine*, if for all $\epsilon > 0$ and all integers $p, q \neq 0$

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha, \epsilon)}{q^{2+\epsilon}}.$$

  By Roth's theorem all algebraic irrationals are Diophantine, and by Khinchin's theorem, almost all $\alpha \in \mathbb{R}$ are Diophantine.

- Conjecture (Rudnick–Sarnak, 1998): Let $d \geq 2$, and let α be a Diophantine irrational. Then the gap distribution of the fractional parts of the sequence $\left(\alpha n^d\right)$ is Poissonian.

- Rudnick–Sarnak's conjecture is open even for the simplest choices of α, e.g., is the gap distribution of the fractional parts of $\left(\sqrt{2}n^2\right)$ Poissonian? We do not even have a "metric" result (i.e., a result holding for almost all α).

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The gaps of $(\sqrt{2}n^2)$

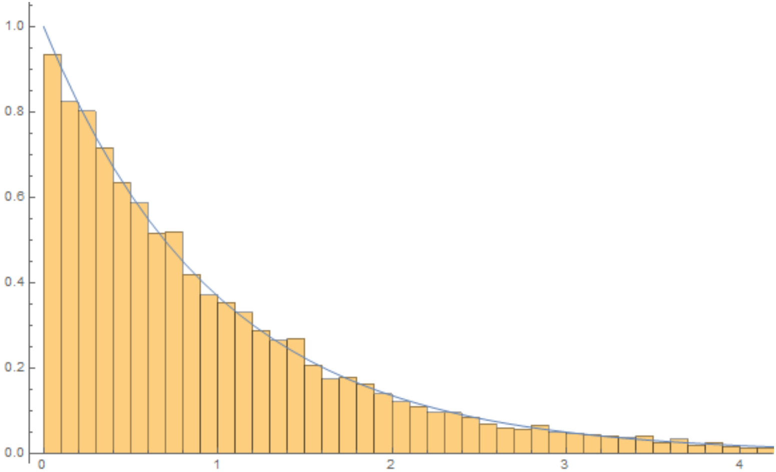

Figure: The gap distribution of the first 10000 fractional parts of the sequence $(\sqrt{2}n^2)$.

The Sequence (n^α)

- Conjecture: Let $\alpha > 0$ such that α in not an integer, and $\alpha \neq 1/2$. Then the gap distribution of the fractional parts of the sequence (n^α) is Poissonian.
- Here again, not a single case is known. The case $\sigma = 1/2$ is special:

Theorem (Elkies–McMullen, 2004)

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The gaps of $(n^{3/4})$

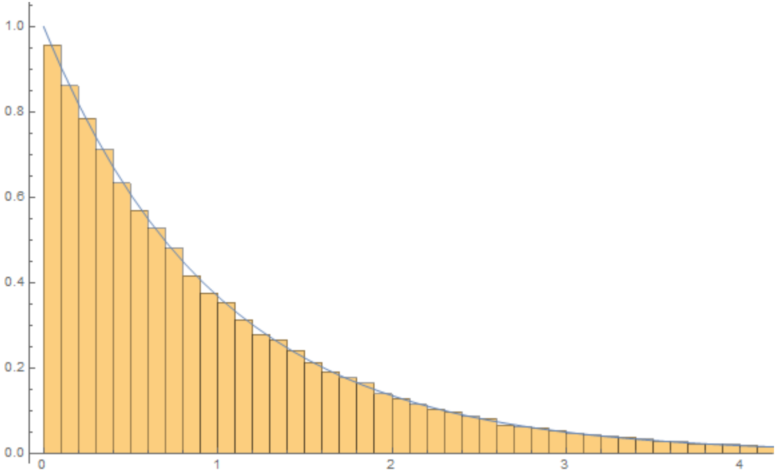

Figure: The gap distribution of the first 10^5 fractional parts of the sequence $(n^{3/4})$.

The gaps of $(\sqrt{n})$

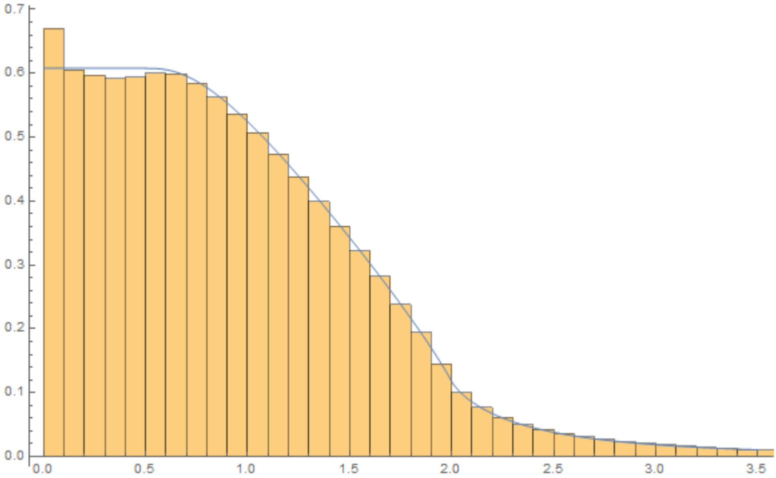

Figure: The gap distribution of the first 10^7 fractional parts of the sequence $(\sqrt{n})$.

Intervals around Points

- Given N points (x_n) in the unit interval and $s > 0$, a random interval of the form $\left[x - \frac{s}{N}, x + \frac{s}{N}\right]$ is expected to contain $2s$ points.
- When the centre of the interval is one of the given points x_m, it contains

$$\#\left\{1 \leq n \leq N : \; n \neq m, \; \|x_n - x_m\| \leq \frac{s}{N}\right\}$$

other points.

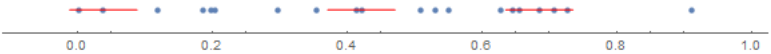

Figure: Counting the number of points in an interval around a point

Pair Correlation

- Averaging over all centres x_m, we get the pair correlation function

$$R_{2,N}([-s,s]) = \frac{1}{N}\#\left\{1 \leq n \neq m \leq N : \|x_n - x_m\| \leq \frac{s}{N}\right\}$$

which measures the average number of (other) points in an interval around a point.

- For N random points (Poisson model), it converges to the Lebesgue measure

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- For $f \in C_c^\infty(\mathbb{R})$ we can equivalently work with the *smooth* pair correlation function

$$R_{2,N}(f) = \frac{1}{N} \sum_{k\in\mathbb{Z}} \sum_{1\leq n\neq m\leq N} f\left(N\left(x_n - x_m - k\right)\right).$$

- Poissonian pair correlation (smooth version):

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Theorem (Rudnick–Sarnak, 1998)

Let $d \geq 2$. For almost all $\alpha \in \mathbb{R}$, the sequence $\left(\alpha n^d\right)$ has Poissonian pair correlation.

- Very little is known for specific values of α. For example, is the pair correlation of $\sqrt{2}n^2$ Poissonian?

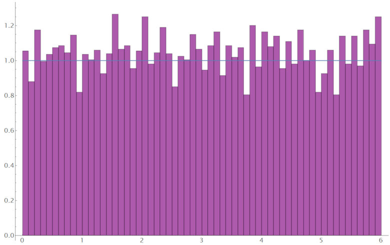
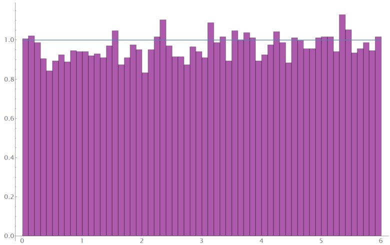


Figure: The pair correlation measure of the first 2000 fractional parts of the sequence $\left(\sqrt{2}n^2\right)$.

The Pair Correlation of $(\sqrt{n})$

Theorem (El-Baz–Marklof–Vinogradov, 2013)

The pair correlation of the sequence $(\sqrt{n})_{\sqrt{n}\notin\mathbb{Z}}$ is Poissonian.



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Higher Correlations

- For $f \in C_c^\infty\left(\mathbb{R}^{k-1}\right)$, the $k-$level correlation is defined by

$$R_{k,N}(f) = \frac{1}{N} \sum_{\mathbf{k}\in\mathbb{Z}^{k-1}} \sum_{\substack{1\le n_1,\ldots,n_k\le N \\ \text{distinct}}} f\left(N\left(\Delta - \mathbf{k}\right)\right)$$

where

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- Poissonian $k-$level correlation:

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The Sequences (n^α) and (α^n)

Theorem (Technau–Y., 2020)

The sequence (n^α) has Poissonian $k-$level correlation for almost all $\alpha > 4k^2 - 4k - 1$.

- This gives Poisson pair correlation for almost all $\alpha > 7$. Can one improve the range of α all the way down to $\alpha > 0$?

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Exponential Sums

- Consider the (smooth) pair correlation $R_{2,N}(f)$ of the sequence $(e^{\alpha a_n})$, where $\alpha \in \mathcal{J} := [A, A+1]$. Applying the Poisson summation formula and using the rapid decay of the Fourier coefficients, we get

$$R_{2,N}(f) = \left(1 - \frac{1}{N}\right) \widehat{f}(0) + \frac{1}{N^2} \sum_{0 \neq |k| \leq N^{1+\epsilon}} \widehat{f}\left(\frac{k}{N}\right) \sum_{1 \leq n \neq m \leq N} e^{2\pi i k \left(e^{\alpha a_n} - e^{\alpha a_m}\right)} + O\left(N^{-\infty}\right)$$

and we would like to show that for almost all $\alpha > 0$, the middle term vanishes as $N \to \infty$.

- By a standard Borel-Cantelli argument, it is enough to show that the variance of $R_{2,N}$ decays with a polynomial rate, which reduces to showing that

$$\frac{1}{N^4}\sum_{\substack{0\neq|k_1|\leq N^{1+\epsilon}\\0\neq|k_2|\leq N^{1+\epsilon}}}\sum_{\substack{1\leq m_1\neq m_2\leq N\\1\leq n_1\neq n_2\leq N}}\left|\int_{\mathcal{J}} e^{2\pi i\phi(\alpha)}\,d\alpha\right| = O\left(N^{-\delta}\right)$$

for some $\delta > 0$, where

$$\phi(\alpha) = k_1\left(e^{\alpha a_{m_1}} - e^{\alpha a_{m_2}}\right) + k_2\left(e^{\alpha a_{n_1}} - e^{\alpha a_{n_2}}\right).$$

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A van der Corput type Lemma

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Let $\phi : \mathcal{J} \to \mathbb{R}$ *be a* C^∞*-function. Fix* $\ell \geq 1$*, and suppose that* $\phi^{(\ell)}(\alpha)$ *has at most* C *zeros, and that the inequality* $\max_{1 \leq i \leq \ell} \left|\phi^{(i)}(\alpha)\right| \geq \lambda > 0$ *holds throughout the interval* $\mathcal{J}$*. If* $\ell = 1$*, assume additionally that* ϕ' *is monotone on* $\mathcal{J}$*. Then*

$$\int_{\mathcal{J}} e^{2\pi i \phi(\alpha)} \, d\alpha \ll_{\ell, C} \lambda^{-1/\ell}.$$

- The crux of the argument is therefore to prove that at each $\alpha \in \mathcal{J}$, at least some derivative of

$$\phi(\alpha) = k_1 \left(e^{\alpha a_{m_1}} - e^{\alpha a_{m_2}}\right) + k_2 \left(e^{\alpha a_{n_1}} - e^{\alpha a_{n_2}}\right)$$

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Repulsion Principle

- To prove repulsion, note that

$$\begin{pmatrix} \phi^{(1)}(\alpha) \\ \phi^{(2)}(\alpha) \\ \phi^{(3)}(\alpha) \\ \phi^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} a_{m_1} & a_{m_2} & a_{n_1} & a_{n_2} \\ a_{m_1}^2 & a_{m_2}^2 & a_{n_1}^2 & a_{n_2}^2 \\ a_{m_1}^3 & a_{m_2}^3 & a_{n_1}^3 & a_{n_2}^3 \\ a_{m_1}^4 & a_{m_2}^4 & a_{n_1}^4 & a_{n_2}^4 \end{pmatrix} \begin{pmatrix} k_1 e^{\alpha a_{m_1}} \\ -k_1 e^{\alpha a_{m_2}} \\ k_2 e^{\alpha a_{n_1}} \\ -k_2 e^{\alpha a_{n_2}} \end{pmatrix}.$$

- Assume for simplicity that $m_1 > m_2 > n_1 > n_2$. Then if we denote the Vandermonde matrix by M, then

$$\max_{1 \le i \le 4} \left| \phi^{(i)}(\alpha) \right| \left\| M^{-1} \right\|_\infty \ge e^{\alpha a_{m_1}}$$

  so that an appropriate lower bound for $\max_{1 \le i \le 4} \left| \phi^{(i)}(\alpha) \right|$ will follow from an upper bound for $\left\| M^{-1} \right\|_\infty$, which we will get from our assumptions.

- There are "degenerate" configurations, e.g., when $m_1 = n_1$, where $\phi(\alpha)$ depends on fewer independent terms. These are handled combinatorially.

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