Zeros of derivatives of the Riemann zeta function and relations to the Riemann hypothesis

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– partly a joint work with Fan Ge –

Kyushu University

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TAU number theory seminar

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Riemann zeta function

The *Riemann zeta function* $\zeta(s)$ is the analytic function on $\mathbb{C}\backslash\{1\}$ satisfying

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{when } \operatorname{Re}(s) > 1. \tag{1}$$

Remarks

- $\zeta(s)$ has a simple pole at $s = 1$ as its only singularity.
- The equality $\sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1}$ tells us that $\zeta(s)$ has no zeros in $\operatorname{Re}(s) > 1$.
- Subtracting the term $(s-1)^{-1}$ from the Dirichlet series (1) and using its integral representation, we find that $\zeta(s)$ can be analytically continued to $\operatorname{Re}(s) > 0$ $(s \neq 1)$.

Functional equation and trivial zeros of $\zeta(s)$

$\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{2}$$

From this we can deduce:

- Since $\zeta(s)$ is analytic on $\mathrm{Re}(s) > 0$ $(s \neq 1)$, $\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ is too.
- At $s = 2, 4, 6, \ldots$, $\sin(\pi s/2) = 0$ cancels out poles of $\Gamma(1-s)$.
- $\zeta(1-s)$ has simple zeros at $s = 3, 5, 7, \ldots$ due to poles of $\Gamma(1-s)$.

Hence $\zeta(s)$ has trivial zeros at $s = -2, -4, -6, -8, -10, \ldots$.

Zeros of $\zeta(s)$

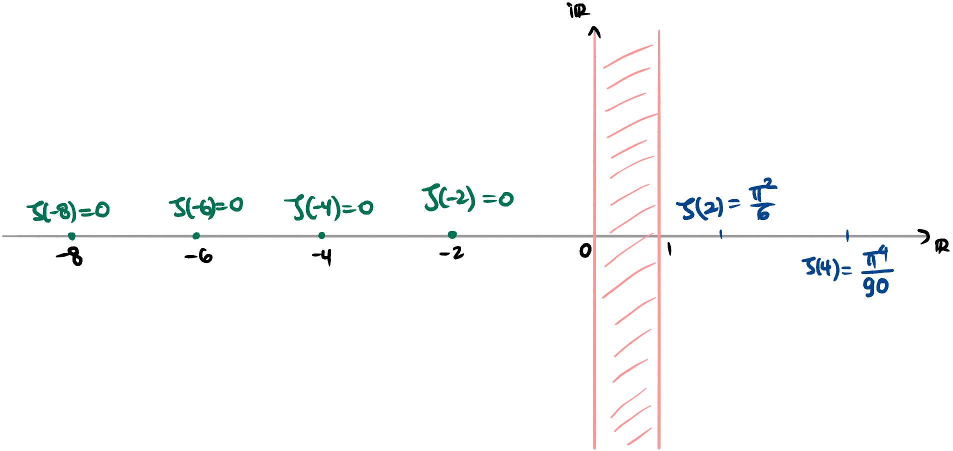
From

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}(s) > 1),$$

we immediately find that $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$.

From the functional equation (2), $\zeta(s) \neq 0$ when $\mathrm{Re}(s) < 0$, except when $s = -2, -4, -6, -8, -10, \ldots$.

Hence, zeros of $\zeta(s)$ other than $s = -2, -4, -6, -8, -10, \ldots$, if exist, should lie within $0 \leq \mathrm{Re}(s) \leq 1$.



iR
ζ(-8)=0
ζ(-6)=0
ζ(-4)=0
ζ(-2)=0
ζ(2)=π²/6
ζ(4)=π⁴/90
-8
-6
-4
-2
0
1
R

The Prime Number Theorem

$$\pi(x) := \#\{p \leq x \mid p \text{ is a prime number}\}, \quad \mathrm{Li}(x) := \int_2^x \frac{1}{\log t}\,\mathrm{d}t$$

Observation (Riemann, 1859)

$$\pi(x) = \mathrm{Li}(x) + (\textit{small order terms})$$

Theorem (Hadamard, de la Vallée Poussin; 1896)

1. $\zeta(1+it) \neq 0, \qquad t > 0$
2. $\pi(x) = \mathrm{Li}(x) + O\left(\dfrac{x}{\exp\left(a\sqrt{\log x}\right)}\right), \qquad \exists a > 0$

Nontrivial zeros of $\zeta(s)$

$\{-2, -4, -6, -8, -10, \cdots\}$ = the set of all *trivial* zeros of $\zeta(s)$

$$\mathcal{Z} := \{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ \rho \notin -2\mathbb{N}\}$$
= the set of all *nontrivial* zeros of $\zeta(s)$

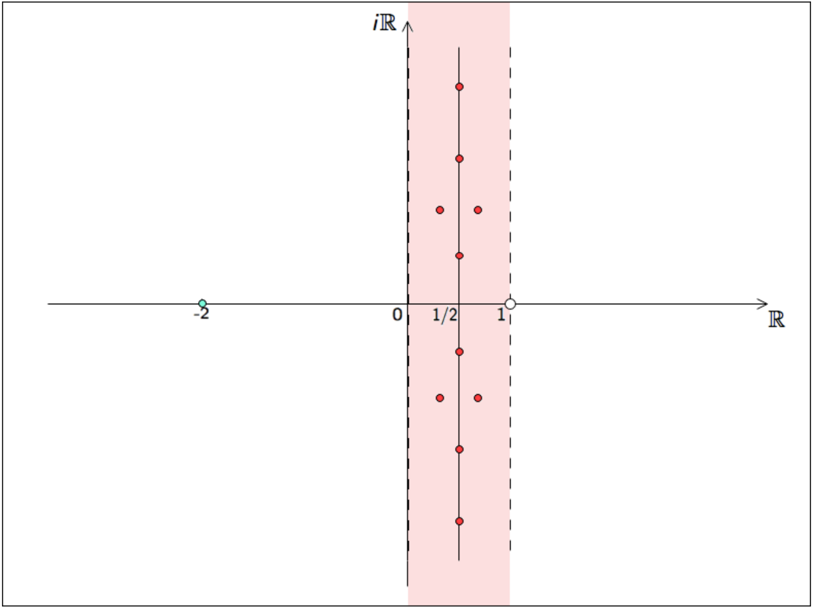
$\rho \in \mathcal{Z}$:

1. $\mathrm{Im}(\rho) \neq 0$,
2. $0 < \mathrm{Re}(\rho) < 1$,
3. $\zeta(\overline{\rho}) = 0$,
4. $\zeta(1 - \overline{\rho}) = 0$.

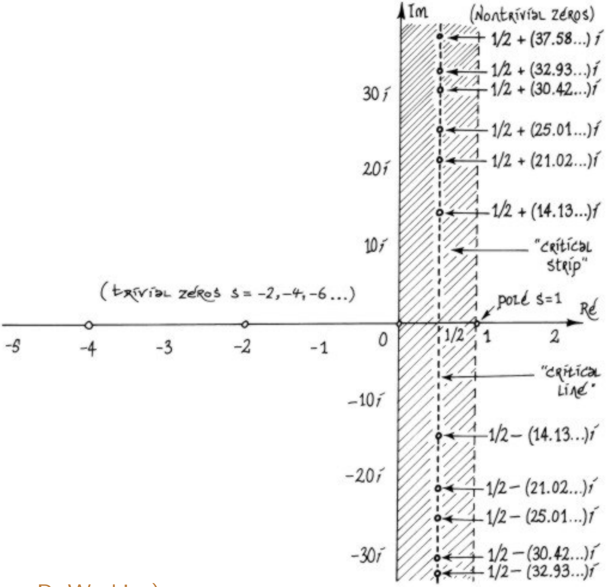
$$\mathcal{Z} = \{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ \mathrm{Im}(\rho) \neq 0\} = \{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ \mathrm{Re}(\rho) > 0\}$$

Riemann hypothesis (RH): For any $\rho \in \mathcal{Z}$, $\mathrm{Re}(\rho) = 1/2$.

Properties of zeros of $\zeta(s)$



Zeros of $\zeta(s)$



(by Matthew R. Watkins)

The Prime Number Theorem

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2. $\pi(x) = \mathrm{Li}(x) + O\left(\dfrac{x}{\exp\left(a\sqrt{\log x}\right)}\right), \qquad \exists a > 0$

Theorem (Koch, 1900)

$$\mathrm{RH} \qquad \Longleftrightarrow \qquad \pi(x) = \mathrm{Li}(x) + O\left(x^{1/2+\epsilon}\right), \quad \forall \epsilon > 0$$

Equivalence for RH

Theorem (Speiser, 1935)

RH

$$\zeta(s) \neq 0 \quad \text{in} \quad 0 < \mathrm{Re}(s) < 1/2$$

is equivalent to

$$\zeta'(s) \neq 0 \quad \text{in} \quad 0 < \mathrm{Re}(s) < 1/2.$$

Theorem (Levinson and Montgomery, 1974)

$N^{-}(T)$ (*resp.* $N_1^{-}(T)$) := the number of zeros of $\zeta(s)$ (*resp.* $\zeta'(s)$) in $\{\sigma + it \mid 0 < \sigma < 1/2,\ 0 < t < T\}$, counted w/ multiplicity.

For $T \geq 2$ we have

$$N^{-}(T) = N_1^{-}(T) + O(\log T).$$

Zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$

Riemann hypothesis (RH): For any $\rho \in \mathcal{Z}$, $\mathrm{Re}(\rho) = 1/2$.

$N(T) :=$ the number of nontrivial zeros ρ of $\zeta(s)$ with $0 < \mathrm{Im}(\rho) < T$, counted with multiplicity

$N_0(T) :=$ the number of zeros $\rho_0 = 1/2 + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$, counted with multiplicity

Riemann hypothesis (RH): $N(T) = N_0(T)$ for all T.

Theorem (Hardy, 1914)

$$N_0(T) \to \infty \quad \text{as} \quad T \to \infty.$$

Theorem (Hardy and Littlewood, 1921)

$$N_0(T) \gg T$$

Simple zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$

Theorem (Selberg, 1942)

There exists $c > 0$ (effective) such that $N_0(T) > c\,N(T)$.

Theorem (Levinson, 1974)

$$N_0(T) \geq 0.3474\,N(T)$$

$N_0^*(T) :=$ the number of zeros $\rho_0 = 1/2 + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ where $\zeta'(\rho_0) \neq 0$

Theorem (Levinson, 1974)

$$N_0^*(T) \geq 0.3474\,N(T)$$

Shifting zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$ by $\zeta'(s)$

- Study the change of argument of

$$G(\frac{1}{2}+it) = \zeta(\frac{1}{2}+it) + \frac{\zeta'(\frac{1}{2}+it)}{\log\frac{t}{2\pi} + O(\frac{1}{|t|})}$$

for large t.

- The number of zeros of $G(s)$ is essentially that of $\zeta(s)$ in $\{\sigma + it \mid 1/2 < \sigma < 1,\ 0 < t < T\}$.

$$\rightsquigarrow \quad \begin{cases} N_G(T) < \frac{1}{3}N(T), \\ \\ N_0(T) = N(T) - 2N_G(T) + O(\log T), \end{cases}$$

$N_G(T) :=$ the number of zeros of $G(s)$ in $\{\sigma + it \mid 1/2 \le \sigma < 3,\ 0 < t < T\}$, counted w/ multiplicity.

Shifting zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$ by $\sum a_k(s)\zeta^{(k)}(s)$

Instead of

$$G(\frac{1}{2} + it) = \zeta(\frac{1}{2} + it) + \frac{\zeta'(\frac{1}{2} + it)}{\log \frac{t}{2\pi} + O(\frac{1}{|t|})},$$

consider

$$g(\frac{1}{2} + it) = \zeta(\frac{1}{2} + it) + \sum_{n \geq 1} a_n(t)\zeta^{(n)}(\frac{1}{2} + it),$$

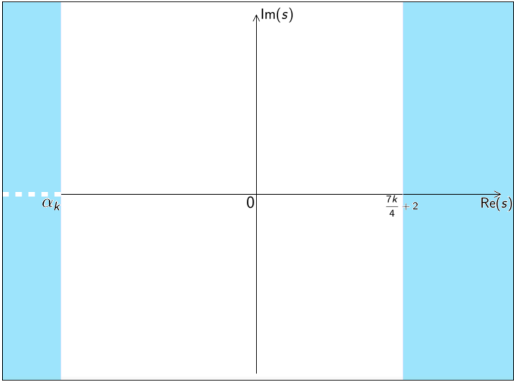
with t large.

Theorem (Conrey, 1989)

$$N_0(T) \geq 0.4088\, N(T), \qquad N_0^*(T) \geq 0.4013\, N(T)$$

Nontrivial (= non-real) zeros of $\zeta^{(k)}(s)$

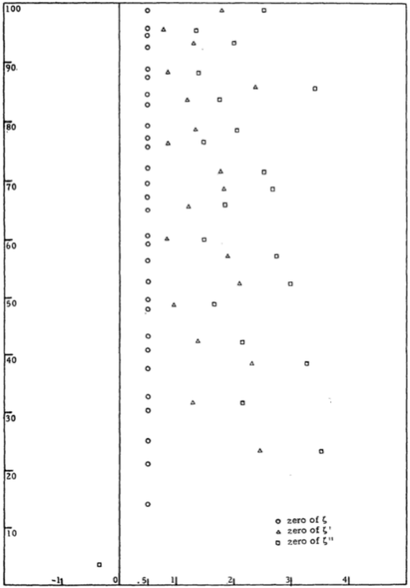
A zero-free region of $\zeta^{(k)}(s)$:



$$\mathcal{Z} = \{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ \ \mathrm{Im}(\rho) \neq 0\}$$

$$\begin{aligned}\mathcal{Z}^{(k)} &:= \{\rho \in \mathbb{C} \mid \zeta^{(k)}(\rho) = 0,\ \ \mathrm{Im}(\rho) \neq 0\} \\ &= \text{the set of all } \textit{nontrivial} \text{ zeros of } \zeta^{(k)}(s)\end{aligned}$$

Nontrivial zeros of $\zeta(s)$, $\zeta'(s)$, $\zeta''(s)$



R. Spira, *Zero-free regions of* $\zeta^{(k)}(s)$,
J. Lond. Math. Soc. **40** (1965), p. 681

RH and zeros of $\zeta''(s)$ & $\zeta'''(s)$

Theorem (Yıldırım, 1996)

RH implies

$$\zeta''(s) \neq 0 \text{ and } \zeta'''(s) \neq 0 \text{ in } 0 \leq \mathrm{Re}(s) < 1/2.$$

Theorem (Yıldırım, 1996)

$\zeta''(s)$ and $\zeta'''(s)$ have only one pair of non-real zeros in $\mathrm{Re}(s) < 0$.

Corollary (Yıldırım, 1996)

RH implies

$\zeta''(s)$ and $\zeta'''(s)$ have only one pair of non-real zeros in $\mathrm{Re}(s) < 1/2$.

RH and non-real zeros of $\zeta^{(k)}(s)$

Theorem (Levinson and Montgomery, 1974)

Let $m \geq 0$.

$\zeta^{(m)}(s)$ has only finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$

$\Rightarrow$

$\zeta^{(m+j)}(s)$ $(j \geq 1)$ also has only finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$.

Corollary (Levinson and Montgomery, 1974)

RH $\Rightarrow$

$\zeta^{(k)}(s)$ has at most finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$.

Theorem 1: Number of nontrivial zeros of $\zeta^{(k)}(s)$ (under RH)

$N(T)$ (*resp.* $N_k(T)$) := the number of nontrivial zeros ρ of $\zeta(s)$ (*resp.* $\zeta^{(k)}(s)$) with $0 < \mathrm{Im}(\rho) < T$, counted with multiplicity

$$g(T) := \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}, \quad h(T) := \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi}$$

	$N(T)$	$N_k(T)$
unconditional	$g(T) + O(\log T)$ [von Mangoldt, 1905]	$h(T) + O_k(\log T)$ [Berndt, 1970]
under RH	$g(T) + O\left(\frac{\log T}{\log\log T}\right)$ [Littlewood, 1924]	$h(T) + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$ $k = 1$: [Akatsuka, 2012] $k \geq 2$: [A.I.S., 2015]

An improvement by Fan Ge (under RH)

$N(T)$ (*resp.* $N_k(T)$) = the number of nontrivial zeros ρ of $\zeta(s)$ (*resp.* $\zeta^{(k)}(s)$) with $0 < \mathrm{Im}(\rho) < T$, counted with multiplicity

$$g(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}, \quad h(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi}$$

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under RH	$g(T) + O\left(\frac{\log T}{\log\log T}\right)$ [Littlewood, 1924]	$h(T) + O\left(\frac{\log T}{\log\log T}\right)$ $k = 1$: [Ge, 2017] $k \geq 2$: XoX

Theorem 2: Improved $N_k(T)$ (under RH)

$N(T)$ (*resp.* $N_k(T)$) $=$ the number of nontrivial zeros ρ of $\zeta(s)$ (*resp.* $\zeta^{(k)}(s)$) with $0 < \mathrm{Im}(\rho) < T$, counted with multiplicity

$$g(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}, \quad h(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi}$$

	$N(T)$	$N_k(T)$
uncon-ditional	$g(T) + O(\log T)$ [von Mangoldt, 1905]	$h(T) + O_k(\log T)$ [Berndt, 1970]
under RH	$g(T) + O\left(\frac{\log T}{\log\log T}\right)$ [Littlewood, 1924]	$h(T) + O_k\left(\frac{\log T}{\log\log T}\right)$ $k = 1$: [Ge, 2017] $k \geq 2$: [Ge and A.I.S., 2020]

Theorem 3: A more general statement (under RH)

Suppose that the error term bound in $N(T)$

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + E_0(T)$$

is $E_0(T) = O(\Phi(T))$ for some increasing function $\log\log T \ll \Phi(T) \ll \log T$.

Theorem 3 (Ge and A.I.S., 2020)

Assume RH. Then

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\max\left\{\Phi(2T), \sqrt{\log T}\log\log T\right\}\right).$$

Counting zeros

Argument principle

f: meromorphic in and on a closed contour $\mathcal{C}$, but has neither zeros nor poles on $\mathcal{C}$,
N (*resp.* P): the number of zeros (*resp.* poles) of f inside $\mathcal{C}$, counted with multiplicity,

$$N - P = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz.$$

Jensen's lemma

Let $f(z)$ be analytic for $|z| < R$ and suppose that $f(0) \neq 0$. Let $n(x)$ denote the number of zeros of $f(z)$ in the disc $|z| \leq x$, then if $r < R$,

$$\int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \log |f(0)|.$$

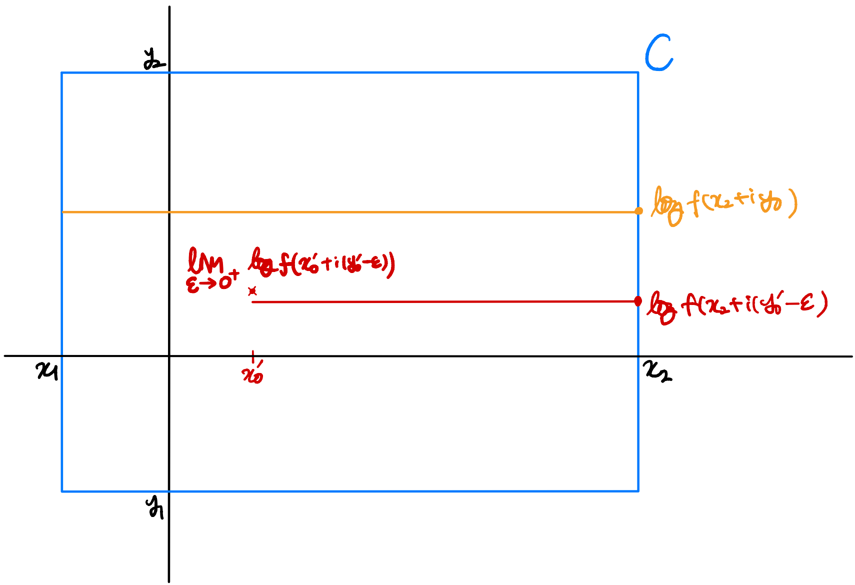
Counting zeros with a distance to a line

Littlewood's lemma

Let C denote the rectangle bounded by the lines $x = x_1$, $x = x_2$, $y = y_1$, and $y = y_2$, where $x_1 < x_2$, $y_1 < y_2$. Let $f(z)$ be analytic and not zero on C, and meromorphic inside it. We define the logarithm $\log f(z)$ by continuous variation along the line $y = y_0$ from $\log f(x_2 + iy_0)$ for $y_1 \leq y_0 \leq y_2$, provided that $[x + iy_0, x_2 + iy_0]$ does not contain any zero or pole of $f(z)$. Otherwise, we put $\log f(z) = \log f(z - i0)$. Then

$$\int_C \log f(z) dz = -2\pi i \sum_{\substack{\beta + i\gamma, \\ f(\beta+i\gamma)=0, \\ x_1 < \beta < x_2, \, y_1 < \gamma < y_2}} (\beta - x_1),$$

where the sum is counted is multiplicity.



C
y_2
y_1
x_1
x_2
x_0'
$\log f(x_2 + i y_0)$
$\lim_{\varepsilon \to 0^+} \log f(x_0' + i(y_0' - \varepsilon))$
$\log f(x_2 + i(y_0' - \varepsilon))$

A preliminary lemma (by argument principle)

Assume RH. Let

$$G_k(s) := (-1)^k \frac{2^s}{(\log 2)^k} \zeta^{(k)}(s)$$

Let $T \geq 2$ satisfy $\zeta(\sigma + iT) \neq 0,\ \zeta^{(k)}(\sigma + iT) \neq 0$ $({}^{\forall}\sigma \in \mathbb{R})$. Then

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi} \arg G_k\left(\frac{1}{2} + iT\right) + \frac{1}{2\pi} \arg \zeta\left(\frac{1}{2} + iT\right) + O_k(1).$$

The arguments are taken such that $\log \zeta(s)$ and $\log G_k(s)$ tend to 0 as $\sigma \to \infty$, and are holomorphic on $\mathbb{C} \backslash \{\rho + \lambda \mid \zeta(\rho) = 0 \textit{ or } \infty,\ \lambda \leq 0\}$ and $\mathbb{C} \backslash \{\rho + \lambda \,|\, \zeta^{(k)}(\rho) = 0 \textit{ or } \infty,\ \lambda \leq 0\}$, respectively.

Sketch of proof

Assume RH. Recall the estimate

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O(1).$$

To simplify we only consider the case when

$$\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) = O\left(\frac{\log T}{\log\log T}\right).$$

Hence taking into acccount

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right) + \frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O_k(1),$$

it suffices to show that

$$\arg G_k\left(\frac{1}{2}+iT\right) = O_k\left(\frac{\log T}{\log\log T}\right).$$

Akatsuka's method

$$\arg G_1(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right), \qquad \frac{1}{2} \le \sigma \le \frac{3}{4},$$

which gives us

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right).$$

Remark

$$\arg G_1(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right), \quad \frac{1}{2} + \frac{(\log\log T)^2}{\log T} \le \sigma \le \frac{3}{4}$$

Fan Ge's method

Write

$$U := \frac{Y}{\log\log T} = \frac{(\log\log T)^2}{\log T}$$

and set

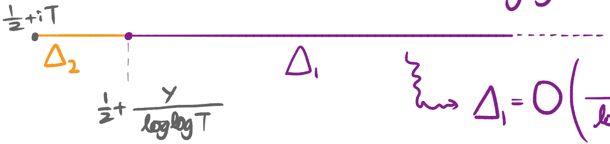
$$\Delta_1 := \underset{\infty+iT\to 1/2+U+iT}{\Delta} \arg G_1(\sigma+iT),$$
$$\Delta_2 := \underset{1/2+U+iT\to 1/2+iT}{\Delta} \arg G_1(\sigma+iT).$$

Then from

$$\arg G_1(\sigma+iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right), \quad \frac{1}{2}+U \le \sigma \le \frac{3}{4},$$

we easily deduce

$$\Delta_1 \ll \frac{\log T}{\log\log T}.$$

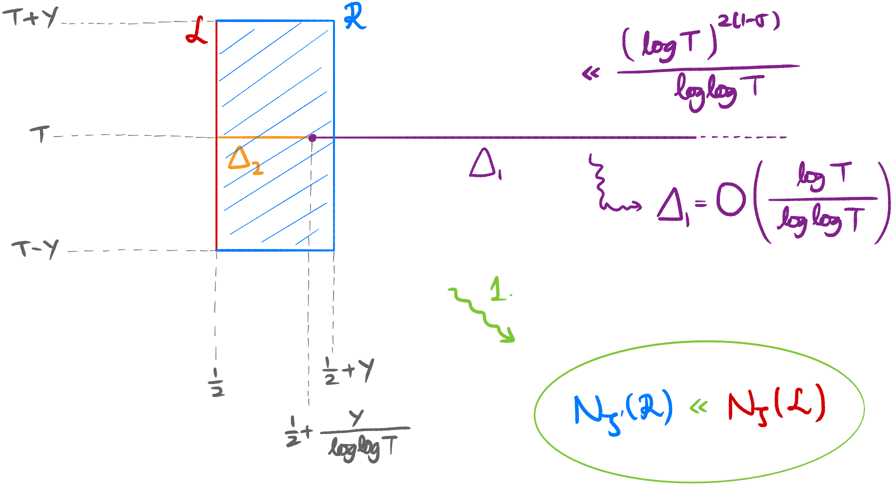


$$\ll \frac{(\log T)^{2(1-\sigma)}}{\log\log T}$$

$$\rightsquigarrow \Delta_1 = O\left(\frac{\log T}{\log\log T}\right)$$

$$\rightsquigarrow \Delta_2 \stackrel{?}{=} O\left(\frac{\log T}{\log\log T}\right)$$

$$Y := \frac{(\log\log T)^3}{\log T}$$



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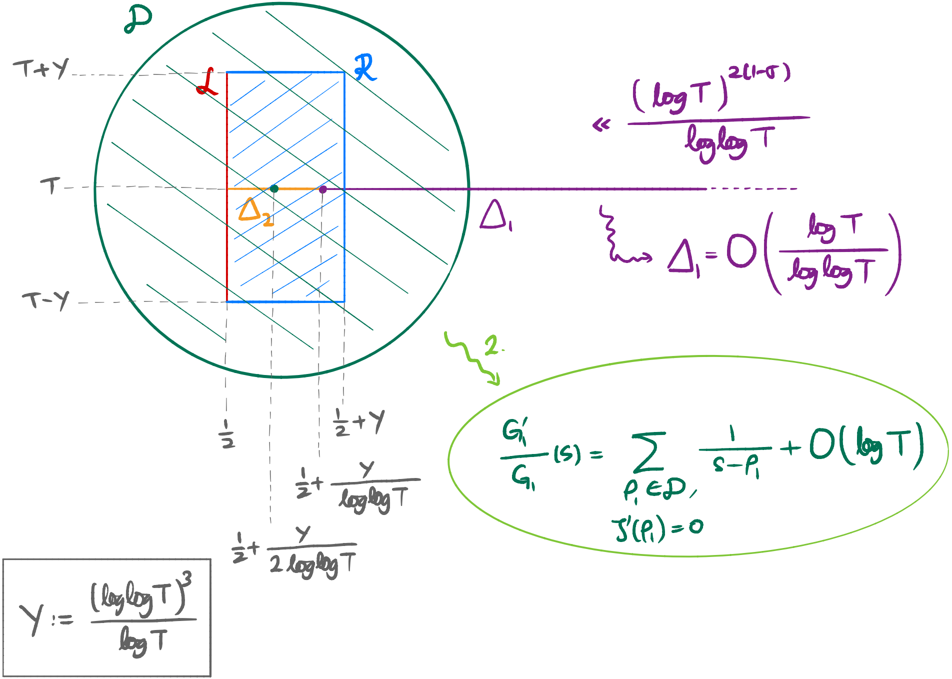


$\mathcal{D}$
$\mathcal{R}$
$\mathcal{L}$
$T+Y$
T
$T-Y$
Δ_2
Δ_1
$\ll \frac{(\log T)^{2(1-\sigma)}}{\log\log T}$
$\Delta_1 = O\left(\frac{\log T}{\log\log T}\right)$
$\frac{1}{2}$
$\frac{1}{2}+Y$
$\frac{1}{2}+\frac{Y}{\log\log T}$
$\frac{1}{2}+\frac{Y}{2\log\log T}$
2.
$\frac{G_1'}{G_1}(s) = \sum_{\rho_1 \in \mathcal{D},\ \zeta'(\rho_1)=0} \frac{1}{s-\rho_1} + O(\log T)$
$Y := \frac{(\log\log T)^3}{\log T}$

Ge's method (continued)

$$
\begin{aligned}
\Delta_2 &= \operatorname{Im} \int_{1/2}^{1/2+U} \frac{G_1'}{G_1}(\sigma + iT)\,d\sigma \\
&= \operatorname{Im} \int_{1/2}^{1/2+U} \sum_{\substack{\rho_1 \in \mathcal{D}, \\ \zeta'(\rho_1)=0}} \frac{1}{\sigma + iT - \rho_1}\,d\sigma + O\left((\log\log T)^2\right) \\
&= \sum_{\substack{\rho_1 \in \mathcal{D}, \\ \zeta'(\rho_1)=0}} \left(\arg\left(\frac{1}{2} + U + iT - \rho_1\right) - \arg\left(\frac{1}{2} + iT - \rho_1\right)\right) \\
&\qquad + O\left((\log\log T)^2\right) \\
&\ll \frac{\log T}{\log\log T}.
\end{aligned}
$$

Extending to higher derivatives – Ge's method

Write

$$X := \frac{1}{\sqrt{\log T}}$$

and set

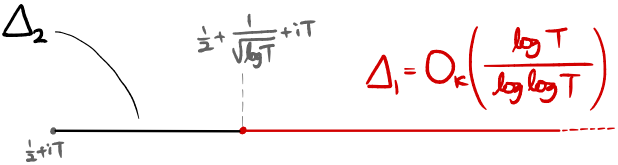
$$\Delta_1 := \underset{\infty+iT\to 1/2+X+iT}{\Delta} \arg G_k(\sigma+iT),$$
$$\Delta_2 := \underset{1/2+X+iT\to 1/2+iT}{\Delta} \arg G_k(\sigma+iT).$$

Then from

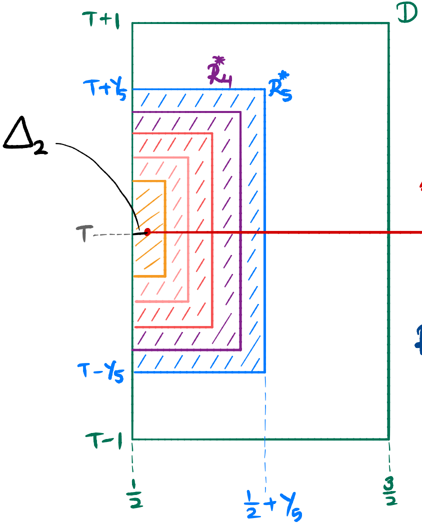
$$\arg \frac{G_k}{\zeta}(\sigma+iT) = O_k\left(\frac{\log\log T}{\sigma-\frac{1}{2}}\right), \quad \frac{1}{2}+\frac{(\log\log T)^2}{\log T} < \sigma < 1$$

we easily deduce

$$\Delta_1 \ll_k \frac{\log T}{\log\log T}.$$



$$\Delta_2 \overset{?}{=} O_\kappa\left(\frac{\log T}{\log\log T}\right)$$



$$\Delta_2 \overset{?}{=} O_\kappa\left(\frac{\log T}{\log\log T}\right)$$

$$\Delta_1 = O_\kappa\left(\frac{\log T}{\log\log T}\right)$$

$$R_j := R_j^* \cup \left\{\tfrac{1}{2}+it \mid T-Y_j \le t \le T+Y_j\right\}$$

$$= \left\{\sigma+it \mid \tfrac{1}{2}\le\sigma\le\tfrac{1}{2}+Y_j,\ T-Y_j\le t\le T+Y_j\right\}.$$

$$Y_j := \frac{2^j}{\sqrt{\log T}}$$

$$\rightsquigarrow \quad N_{\zeta^{(\kappa)}}(R_j) \ll_\kappa Y_j \log T + \frac{\log T}{\log\log T}$$

Tak 感謝

Dziękuję

감사합니다 teşekkür ederim

cảm ơn bạn Баярлалаа

Vielen Dank

Labai ačiū Obrigado

Matur nuwun

Спасибо ขอบคุณค่ะ धन्यवाद् நன்றி Kiitos

谢谢 ধন্যবাদ Thank you Dankjewel شكرا

Gracias Ευχαριστώ Tack

Terima kasih

Хвала вам Merci بہت شکریہ

Köszönöm

ありがとうございます

תודה Takk

Salamat po 多謝

Grazie

keep updating . . .