EXERCISE 1
NUMBER THEORY SEMINAR 2011
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Show that the ABC theorem is a special case of the Theorem of Brownawell-Masser and of Voloch.

Explanation: For a polynomial $A(t) = c\prod_{j=1}^{n}(t-\alpha_j)^{k_j} \in \mathbb{C}[t]$ of positive degree, with $\alpha_j \in \mathbb{C}$ distinct and $k_j \geq 1$, the radical is

$$\operatorname{rad}(A) := \prod_{j=1}^{n}(t-\alpha_j)$$

The polynomial ABC conjecture (Mason, Stothers),which we proved in class, states that if $A, B, C, \in \mathbb{C}[t]$ are coprime polynomials, not all constants, and $A + B = C$, then

$$\max(\deg A, \deg B, \deg C) \leq \deg \operatorname{rad}(ABC) - 1$$

We also stated a theorem of Brownawell-Masser and Voloch, asserting that if $S \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is a finite set of points (including the point at infinity), and $u_1, \dots, u_m \in \mathbb{C}(t)$ are linearly independent, non-constant "S-units", that is rational functions all of whose zeros and poles lie in S, so that $u_1 + \cdots + u_m = 1$, then their heights satisfy

$$\max_{j=1,\dots,m} \operatorname{Ht}(u_j) \leq \frac{m(m-1)}{2}\{\#S - 2\}$$

Here the height $\operatorname{Ht}(u)$ of a rational function $u \in \mathbb{C}(t)$ is the number of its zeros (with multiplicities), and equals the number of its poles (with multiplicities); in both counts one includes the point at infinity, and the order of the zero/pole of $u(t)$ at ∞ is defined as the order of zero/pole of $u(1/t)$ at $t = 0$. So for example if $u(t)$ is a polynomial of degree n then it has n zeros (in the finite plane), and a pole of order n at infinity.

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