Quantum chaos and random surfaces of large genus

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Quantum Unique Ergodicity

- X : compact negatively curved Riemann surface.
- $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$: eigenvalues of the Laplacian on X.
- $\{\psi_{\lambda_j}\}_{j\in\mathbb{N}}$: orthonormal basis of $L^2(X)$ of eigenfunctions of the Laplacian with eigenvalues λ_j.

Conjecture (Rudnick, Sarnak, 1990s)

If $a : X \to \mathbb{C}$ is continuous, then

$$\lim_{j\to\infty} \int_X a(x)|\psi_{\lambda_j}(x)|^2 \, dx = \int_X a(x)\frac{1}{|X|} \, dx.$$

Examples?

- Do there exist X that satisfies QUE?
- Is there a **random model** of compact negatively curved surfaces X, where with positive probability QUE holds?

Graphs

Let $G = (V, E)$ be a d-regular graph.

- **Laplacian on** G: $\Delta_G f(x) = \frac{1}{d} \sum_{y \in V, y \sim x} f(y) - f(x)$.
- **Uniform probability model** $\mathbb{P}_{N,d}$ for the space $\mathcal{G}_{N,d}$ of d-regular graphs with $|V| = N$ is given by $\mathbb{P}_{N,d}(\mathcal{A}) = \frac{|\mathcal{A}|}{|\mathcal{G}_{N,d}|}, \quad \mathcal{A} \subset \mathcal{G}_{N,d}$.
- $\mathcal{E}_G$: set of all L^2 normalized eigenfunctions on G.

Theorem (Bauerschmidt, Knowles, Yau 2017)

Let G be a $\mathbb{P}_{N,d}$ random regular graph. Then for any $\psi_\lambda \in \mathcal{E}_G$ with eigenvalue λ we have

$$\|\psi_\lambda\|_\infty \le C(\lambda) \frac{(\log N)^2}{\sqrt{N}}$$

with probability at least $1 - e^{-2(\log N)^3}$ as long as $d \ge (\log N)^4$.

Note: $\| \frac{1}{\sqrt{N}} \|_\infty = \frac{1}{\sqrt{N}}$ and $\| \frac{1}{\sqrt{N}} \|_2 = 1$.

Surfaces?

$$\text{“large cardinality } |V| \quad \leftrightarrow \quad \text{large area } |X|\text{”}$$

For arithmetic surfaces there is the **level aspect theory**.

Example:

Theorem (Saha 2014)

Fix a level $n \in \mathbb{N}$ and let $X := \Gamma_0(n) \setminus \mathbb{H}$, where

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod n \right\}.$$

If ψ_λ is L^2-normalized Hecke-Maass cuspidal newform on X of Laplacian eigenvalue λ, then for all $\varepsilon > 0$ we have

$$\|\psi_\lambda\|_\infty \leq C(\lambda, \varepsilon)|X|^{-1/12+\varepsilon}.$$

Random surfaces

- $\mathcal{M}_g$: moduli space of compact hyperbolic surfaces of genus g.
- $\mathbb{P}_g$: uniform probability on $\mathcal{M}_g$ w.r.t. Weil-Petersson volume.
- $\mathcal{E}_X$: set of all L^2 normalized eigenfunctions on X.

Theorem (Gilmore, Le Masson, S., Thomas 2019)

Let X be a $\mathbb{P}_g$ random hyperbolic surface of genus g. Then for any $\varepsilon > 0$ and $\psi_\lambda \in \mathcal{E}_X$ with eigenvalue $\lambda \geq 1/4$ we have

$$\|\psi_\lambda\|_\infty \leq C(\lambda, \varepsilon) \frac{1}{\sqrt{\log |X|}}$$

with probability at least $1 - O(g^{-1/2+\varepsilon})$.

Note: $\|\frac{1}{\sqrt{|X|}}\|_\infty = \frac{1}{\sqrt{|X|}}$ and $\|\frac{1}{\sqrt{|X|}}\|_2 = 1$.

Possible future work

Conjecture

Let X be a $\mathbb{P}_g$ random hyperbolic surface of genus g. Then for any $\varepsilon > 0$ and $\psi_\lambda \in \mathcal{E}_X$ with eigenvalue $\lambda > 1/4$ we have

$$\|\psi_\lambda\|_\infty \leq C(\lambda) \frac{(\log |X|)^{\alpha(\varepsilon)}}{\sqrt{|X|}}$$

with probability tending to 1 as $g \to \infty$, where $\alpha(\varepsilon) > 0$ is some function.

Theorem (Bauerschmidt, Knowles, Yau 2017)

Let G be a $\mathbb{P}_{N,d}$ random regular graph. Then for any $\psi_\lambda \in \mathcal{E}_G$ with eigenvalue λ we have

$$\|\psi_\lambda\|_\infty \leq C(\lambda) \frac{(\log N)^2}{\sqrt{N}}$$

with probability at least $1 - e^{-2(\log N)^3}$ as long as $d \geq (\log N)^4$.

L^p norms

Theorem (Gilmore, Le Masson, S., Thomas 2019)

Let X be a $\mathbb{P}_g$ random hyperbolic surface of genus g. Then for any $\varepsilon > 0$ and $\psi_\lambda \in \mathcal{E}_X$ with eigenvalue $\lambda \geq 1/4$ and $p \geq 2 + 4\sqrt{\max\{\frac{1}{4} - \lambda_1(X), 0\}}$ we have

$$\|\psi_\lambda\|_p \leq C(p, \lambda, \varepsilon) \frac{1}{\sqrt{\log |X|}}$$

with probability at least $1 - O(g^{-1/2+\varepsilon})$.

Proof structure

- $N_r(X)$: maximal number of primitive geodesic loops of length $\leq r$ passing through a point in X.
- **Short Loop Condition**: suppose there exists $R(X) \geq 0$ such that for any $r \leq R(X)$ and $\delta > 0$ we have

$$N_r(X) \leq C(\delta) e^{\delta r}.$$

- Under the S.L.C. above, study convolution operators with smoothened cosine wave kernels originating in the work of Iwaniec-Sarnak (1995) and apply Selberg transform to prove

$$\|\psi_\lambda\|_\infty \leq C(\lambda) \frac{\|\psi_\lambda\|_2}{\sqrt{R(X)}}.$$

- Prove an effective version of Mirzakhani-Petri (2017), which gives us: $\exists \delta > 0$ s.t. $\forall c > 0$:

$$\mathbb{P}_g(X : R(X) \geq c \log |X|) \geq 1 - O(g^{-1/2+\delta c})$$

Proof of $\|\psi_\lambda\|_\infty \leq C(\lambda)\frac{\|\psi_\lambda\|_2}{\sqrt{R(X)}}$

- Short Loop Condition is trivially satisfied with $R(X) = \mathrm{InjRad}(X)$.
- $R(X) = \mathrm{InjRad}(X)$ case is a quite direct application of Selberg pre-trace formula (next slide)
- **However**: Mirzakhani (2013) proved $\mathbb{P}_g$ random X has $\mathrm{InjRad}(X)$ uniformly bounded in g with positive probability

Proof of $\|\psi_\lambda\|_\infty \leq C(\lambda)\frac{\|\psi_\lambda\|_2}{\sqrt{R(X)}}$ with $R(X) = \mathrm{InjRad}(X)$

- Let ψ_λ be L^2 normalized and $\Delta\psi_\lambda = \lambda^2\psi_\lambda$, $X = \mathbb{H}/\Gamma$
- Let $R = R(X)$ and $\chi_R(s) = R\chi(Rs)$ where $\chi : \mathbb{R} \to \mathbb{R}_+$ is smooth even rapid decaying with $\mathrm{supp}(\widehat{\chi}) \subset [-1,1]$, $\int \chi = 1$ (so $\chi_R(0) \approx R$)
- **Selberg pre-trace formula**:

$$\sum_{j=1}^{\infty} \chi_R(\lambda_j - \lambda)|\psi_{\lambda_j}(z)|^2 = \frac{1}{2\pi}\int_0^\infty \chi_R\Big(\sqrt{\frac{1}{4} + r^2} - \lambda\Big) r \tanh(\pi r)\, dr$$

$$+ \sum_{\gamma \in \Gamma - \{\mathrm{id}\}} k_{R,\lambda}(d(z, \gamma z))$$

where $k_{R,\lambda}(\varrho)$ is the radial kernel of

$$\int_0^\infty e^{-it\lambda}\widehat{\chi_R}(t)\, \cos(t\sqrt{\Delta})\, dt.$$

- $\mathrm{supp}(k_{R,\lambda}) \subset [-R, R]$ so $\sum_{\gamma\in\Gamma-\{\mathrm{id}\}} = 0$, so by $\chi_R(t) = O((Rt)^{-2})$

$$R|\psi_\lambda(z)|^2 \lesssim \frac{s}{2\pi}\tanh(\pi s) + O(R^{-2}), \quad s = \sqrt{\lambda^2 - \frac{1}{4}}$$

Proof of $\|\psi_\lambda\|_\infty \leq C(\lambda)\frac{\|\psi_\lambda\|_2}{\sqrt{R(X)}}$: general case

- Key: $\cos(t\sqrt{\Delta})$ has finite speed of propagation (kernel non-zero only when $d(x,y) \leq t$).
- Essentially replace $\cos(t\sqrt{\Delta})$ by $\frac{\cos(t\sqrt{\Delta})}{\sqrt{\cosh(\frac{\pi}{2}\sqrt{\Delta})}}$
- Has rapid enough decay outside balls of radius $4t$ and also exponential L^∞ bounds
- For L^p norms we need to employ a TT^* argument.

Geometric side

Enough to prove: there exists $\delta > 0$ s.t. for all $c > 0$

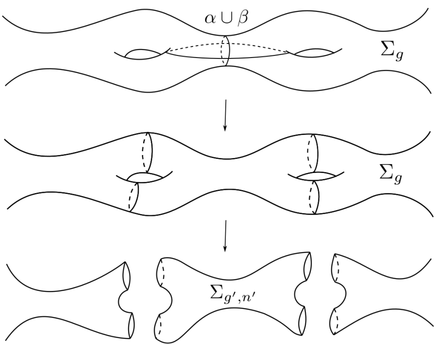
$$\mathbb{P}_g(\mathcal{A}_c) = O(g^{-1/2+\delta c}),$$

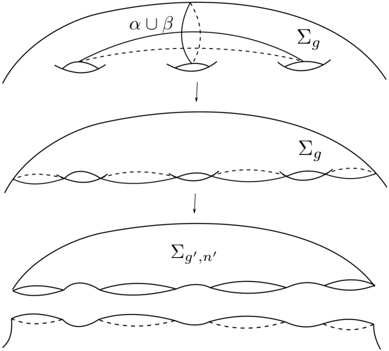
where

$$\mathcal{A}_c = \left\{ X \in \mathcal{M}_g \;\middle|\; \begin{array}{l} \text{There exists } x \in X \text{ such that there are} \\ \text{at least two primitive geodesic loops } \gamma \text{ in } X \\ \text{of length } \ell(\gamma) \leq c\log(g) \text{ passing through } x \end{array} \right\}$$

Geometric side: Simple geodesics case

- Pick $X \in \mathcal{A}_c$ and pick two such loops α, β of length $\leq c \log g$.
- Assume α, β are **simple**
- Then we can extract a separating multicurve $\Gamma = (\gamma_1, \dots, \gamma_4)$ of 4 components with total length $\leq 4c \log g$.





- In general: if geodesics are not simple, Γ has $O(g^{2c})$ curves.

To Weil-Petersson integrals

Markov inequality: if $\mathcal{F} : \mathcal{M}_g \to \mathbb{R}_+$ is $\mathbb{P}_g$ integrable, then

$$\mathbb{P}_g(\mathcal{F}(X) > t) \leq \frac{1}{t} \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} \mathcal{F}(X)\, dX$$

Moduli space notation

$S_{g,n}$: topological surface of genus g with n boundary components

- $\mathcal{T}(S_{g,n})$: Teichmüller space of $S_{g,n}$
- $\mathrm{Mod}(S_{g,n})$: Mapping class group of $S_{g,n}$
- $\mathcal{M}_{g,n} = \mathcal{T}(S_{g,n})/\mathrm{Mod}(S_{g,n})$ moduli space of $S_{g,n}$

Compact case: $S_g = S_{g,0}$, $\mathcal{M}_g = \mathcal{M}_{g,0}$

Applying Weil-Petersson volume estimates

- Define $F : \mathbb{R}_+^4 \to \mathbb{R}_+$ by
$$F(x) = \mathbf{1}(x_1 + \cdots + x_4 \leq 4c \log g)$$
and for $X \in \mathcal{M}_g$
$$F^\Gamma(X) = \sum_{[\alpha_i] \in \mathrm{Mod}(S_g)\cdot\Gamma} F(\ell_X(\alpha_1), \ldots, \ell_X(\alpha_4)).$$
- **Mirzakhani's integration formula**:
$$\int_{\mathcal{M}_g} F^\Gamma(X)\, dX = C_\Gamma \int_{\mathbb{R}_+^4} F(x) V_{g,n}(\Gamma, x)\, x_1 \ldots x_4 dx_1 \ldots dx_4$$

Here

- $V_{g,n}(\Gamma, x)$: Volume of the moduli space $\mathcal{M}(S_{g,n}(\Gamma), \ell_\Gamma = x)$
- $S_{g,n}(\Gamma)$: surface with $n+4$ boundary components obtained by cutting $S_{g,n}$ along the 4 curves of the multicurve Γ

Then using quantitative estimates for $V_{g,n}(\Gamma, x)$ (in terms of g and n):
$$\lim_{g\to\infty} \sum_\Gamma \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} F^\Gamma(X)\, dX = 0.$$