EXERCISE 4
OPEN PROBLEMS IN NUMBER THEORY 2017/18
DUE DATE: MAY 16, 2018

Exercise 1. *The Gaussian* $g(x) = e^{-\pi x^2}$ *lies in* $\mathcal{S}(\mathbb{R})$. *Show that the Fourier transform of* g *is itself:* $\widehat{g} = g$. *Here the Fourier transform is normalized as* $\widehat{g}(x) := \int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}dy$.

Exercise 2. *Show that there are exactly* 6 *different representations of* 4^a *as a sum of three integer squares:* $r_3(4^a) = 6$.

Exercise 3. *Jarnik's theorem states that there is some* $c > 0$ *so that any arc of length* $< cR^{1/3}$ *on the circle* $x^2 + y^2 = R^2$ *of radius* R *cannot contain more than two lattice points. Show that the three lattice points*

$$(4n^3 - 1, 2n^2 + 2n), \quad (4n^3, 2n^2 + 1), \quad (4n^3 + 1, 2n^2 - 2n)$$

all lie on the circle $x^2 + y^2 = R_n^2$, *with* $R_n^2 = 16n^6 + 4n^4 + 4n^2 + 1$, *and are contained in an arc of length* $(16R_n)^{1/3} + o(1)$. *Hence the exponent* $1/3$ *in Jarnik's theorem is sharp.*