Exercise 1. Let
\[ N_3(R) = \# \mathbb{Z}^3 \cap B(0, R) \]
be the number of lattice points in the 3-dimensional ball of radius \( R \). Show that
\[ N_3(R) = \frac{4\pi}{3} R^3 + O(R^{3/2}). \]

Exercise 2. Let \( N \gg 1 \). Show that for any triple \( \{d_1, d_2, d_3\} \) of divisors of \( N \), so that \( \sqrt{N} \leq d_1 < d_2 < d_3 \leq N \), we must have \( d_3 - d_1 \geq N^{1/6} \). That is, the three integer points \( (d_j, N/d_j) \) on the hyperbola \( xy = N \) cannot lie in an arc of diameter \( \ll N^{1/6} \).

Hint: One way to do this is to prove that
\[ \prod_{1 \leq i < j \leq 3} (d_j - d_i) \geq N^{1/2}. \]

Exercise 3. Here is an example showing why one needs the exponent to be bigger than 1 in the abc conjecture: Let \( A_k = 3^{2^k} - 1 \), \( B_k = 1 \), \( C_k = 3^{2^k} \). Show that for all \( k \geq 2 \),
\[ C_k \geq \frac{2^{k+1}}{3} \text{rad}(A_k B_k C_k) \]
and hence that
\[ C_k \gg R_k \log R_k, \]
where \( R_k := \text{rad}(A_k B_k C_k) \).

Hint: Show that \( 2^{k+2} \) divides \( 3^{2^k} - 1 \).

Exercise 4. Show that the abc conjecture implies Pillai’s conjecture: each positive integer occurs at most finitely many times as a difference of perfect powers, i.e. for each \( k \geq 1 \), there are at most finitely many solutions of \( x^m - y^n = k \) with \( x, y, m, n \geq 2 \).

Exercise 5. Prove the polynomial Catalan conjecture: There are no consecutive perfect powers of positive degree in \( \mathbb{C}[x] \), i.e. no solutions of \( f(x)^m - g(x)^n = 1 \) with \( f, g \in \mathbb{C}[x], m, n \geq 2 \) and \( \deg f > 0 \).