PRIMES IN ARITHMETIC PROGRESSIONS: VARYING MODULUS

1. Discussion on PNT in arithmetic progressions with varying modulus

The prime number theorem for arithmetic progressions gives for fixed q

$$\pi(x;q,a) \sim \frac{x}{\phi(q)\log x}. \tag{1}$$

It is much more difficult and interesting to establish this asymptotic for q that grows in terms of x, and this is often crucial for applications. In this direction the Siegel-Walfisz theorem states that for $q \leq (\log x)^A$ (1) holds. The Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions implies that this is true for $q \leq x^{1/2-o(1)}$.

A conjecture of Hugh Montgomery predicts that the asymptotic should hold in even a greater range $q \leq x^{1-o(1)}$ [1], and work of Friedlander and Granville [1] shows that this is essentially best possible. This should be compared to the distribution of primes in short intervals and of the work of Maier [6], which we have previously discussed.

Although GRH is still open we can say quite a bit more about the remainder term

$$E(x;q,a) := \psi(x;q,a) - \frac{x}{\phi(q)}$$

with uniformity on q, on average. First note that this is only interesting when $q < x$ since for $q > x$ there are not many primes $\leq x$ in progressions modulo q. The Barban-Davenport-Halberstam-Montgomery-Hooley theorem see [8] and [4] states that

$$\frac{1}{Q}\sum_{q\leq Q}\sum_{\substack{a \,(\mathrm{mod}\, q)\\ \gcd(a,q)=1}} |E(x;q,a)|^2 \sim x\log Q$$

for $x/(\log x)^A < Q < x$ and on GRH for $x^{1/2+o(1)} < Q < x$. Assuming a conjecture on the second moment of the one level density of zeros of Dirichlet L-functions it follows that this holds for $x^{o(1)} < Q < x$.

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[1] Montgomery's conjecture states $\psi(x;q,a) = x/\varphi(q) + O(x^{o(1)}(x/q)^{1/2})$ (this version of the conjecture was first given by Friedlander and Granville [1]). In the original formulation (see [9]) of the conjecture the error term was stated as $(x/\varphi(q))^{1/2+o(1)}\log x$ and this was proven to be false.

Another result in this direction is the famous theorem of Bombieri and Vinogradov which asserts that for any $A \geq 1$ and $Q < x^{1/2}/(\log x)^B$, for $B = B(A)$,

$$\frac{1}{Q}\sum_{q \leq Q} \max_{\substack{a \pmod q \\ \gcd(a,q)=1}} |E(x;q,a)| \ll \frac{x}{Q(\log x)^A}$$

It may be the case that even moreis true and it has been conjectured by Elliot and Halberstam that for $Q < x^{1-o(1)}$

$$\frac{1}{Q}\sum_{q \leq Q} \max_{\substack{a \pmod q \\ \gcd(a,q)=1}} |E(x;q,a)| \ll \frac{x}{Q(\log x)^A}.$$

Friedlander and Granville [1] showed that this does not hold for $Q = x$.

2. The Brun Titchmarsh Inequality

In a different direction the Brun-Titchmarsh theorem gives an upper bound for the number of primes congruent to $a \pmod q$ with great uniformity in q.

Theorem 2.1 (Brun-Titchmarsh Inequality)**.** *Let a, q be integers with $\gcd(a,q) = 1$ and suppose that $q = o(x)$. Then*

$$\pi(x;q,a) \leq \frac{2x}{\varphi(q)\log x/q}\left(1 + o(1)\right).$$

Remark. *Using the large sieve, Montgomery and Vaughan have shown that the $1 + o(1)$ factor on the RHS of the above inequality can be removed.*

We can rewrite the RHS as

$$C\frac{x}{\varphi(q)\log x}$$

with

$$C = \frac{2}{1 - \frac{\log q}{\log x}}.$$

Improving the value of C is a problem that has been studied by several authors including Motohashi [10], Goldfeld [3], Iwaniec [5], Friedlander and Iwaniec [2], and Maynard [7]. Establishing a version of Brun-Titchmarsh with $C < 2$ would have important consequences.

We first require the following lemma

Lemma 2.2. *For any integer q*

$$\sum_{\substack{n \leq z \\ \gcd(n,q)=1}} \frac{\mu^2(n)}{\varphi(n)} \geq \frac{\varphi(q)}{q}\log z.$$

Proof. The strategy is to compare

$$\sum_{\substack{n\le z\\ \gcd(n,q)=1}} \frac{\mu^2(d)}{\varphi(d)} \qquad \text{to} \qquad \sum_{n\le z} \frac{\mu^2(d)}{\varphi(d)}.$$

the later sum is easily seen to be bounded below by $(\zeta(2))^{-1}\log z$ and with a bit more effort the $(\zeta(2))^{-1}$ factor can be removed. Observe that

$$\begin{aligned}
\sum_{n\le z} \frac{\mu^2(d)}{\varphi(d)} &= \sum_{\ell | q} \sum_{\substack{n\le z\\ \gcd(n,q)=\ell}} \frac{\mu^2(d)}{\varphi(d)} \\
&= \sum_{\ell|q} \sum_{\substack{h\le z/\ell\\ \gcd(h,q/\ell)=1, \gcd(h,\ell)=1}} \frac{\mu^2(h\ell)}{\varphi(h\ell)} \\
&= \sum_{\ell|q} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{h\le z/\ell\\ \gcd(h,\ell)=1}} \frac{\mu^2(h)}{\varphi(h)} \\
&\le \sum_{\ell|q} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{h\le z\\ \gcd(h,\ell)=1}} \frac{\mu^2(h)}{\varphi(h)}.
\end{aligned}$$

To complete the proof note that

$$\sum_{\ell|q} \frac{\mu^2(\ell)}{\varphi(\ell)} = \prod_{p|q}\left(1+\frac{1}{p-1}\right) = \prod_{p|q}\left(1-\frac{1}{p}\right)^{-1} = \frac{q}{\varphi(q)}.$$

□

Proof of Brun-Titchmarsh Inequality. We want to bound

$$\pi(x;q,a) = \#\{p\le x : p\equiv a \pmod q\}.$$

Let

$$\mathcal{A} = \{n\le x : n\equiv a\,(\mathrm{mod}\ q)\}$$

and $\mathcal{P} = \{p : \gcd(p,q)=1\}$. Our key observation is that if $p' \in \{p\le x : p\equiv a \pmod q\}$ then $p' \in \{n\le x : n\equiv a \pmod q, \gcd(n,P(z))=1\}$ or $p'\in\{p\le z\}$ so that

$$\pi(x;q,a) \le \#\{n\le x : n\equiv a \pmod q, \gcd(n,P(z))=1\} + z. \tag{2}$$

Write $\mathcal{S}(\mathcal{A},\mathcal{P},z)$ for the first term on the RHS of the above inequality.

For each d such that if $p|d$ then $p\in\mathcal{P}$

$$\mathcal{A}_d = \{n\in\mathcal{A} : d|n\}.$$

Observe that

$$\#\mathcal{A}_d = \sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\ q),d|n}} 1 = \sum_{\substack{\ell\le x/d\\ d\ell\equiv a\,(\mathrm{mod}\ q)}} 1$$

Since if $p|d$ then $p\in\mathcal{P}$ we know $\gcd(d,q)=1$ so that d is invertible modulo q. Writing $\overline{d}$ for the multiplicative inverse of d modulo q (i.e. $d\overline{d}\equiv 1 \pmod q$) we have

$$\sum_{\substack{\ell\le x/d\\ d\ell\equiv a\,(\mathrm{mod}\ q)}} 1 = \sum_{\substack{\ell\le x/d\\ \ell\equiv \overline{d}a\,(\mathrm{mod}\ q)}} 1.$$

The condition $\ell\equiv\overline{d}a \pmod q$ implies we can write $\ell = qm+r$ with $r\equiv\overline{d}a \pmod q$ and $|r|<q$. Thus,

$$\sum_{\substack{\ell\le x/d\\ \ell\equiv \overline{d}a\,(\mathrm{mod}\ q)}} 1 = \sum_{m:qm+r\le x/d} 1 = \frac{x}{qd}+O(1).$$

Note that $X := \#\mathcal{A} = \frac{x}{q}+O(1)$. Hence, we can write

$$\#\mathcal{A}_d = \frac{X}{f(d)}+R_d$$

with $f(d)=d$, $X=x/q+O(1)$ and $R_d=O(1)$. Therefore the Selberg sieve gives

$$\mathcal{S}(\mathcal{A},\mathcal{P},z)\le\frac{X}{S(z)}+R(z)$$

where, by Lemma 2.2

$$S(z)=\sum_{\substack{d\le z\\ d|P(z)}}\frac{\mu^2(d)}{(\mu*f)(d)}=\sum_{\substack{d\le z\\ \gcd(d,q)=1}}\frac{\mu^2(d)}{\varphi(d)}\ge\frac{\varphi(q)}{q}\log z$$

and

$$R(z)=\sum_{\substack{d_1,d_2\le z\\ d_1,d_2|P(z)}}|R_{[d_1,d_2]}|\ll z^2.$$

Thus, by these estimates along with (2) we have

$$\pi(x;q,a)\le\frac{x}{\varphi(q)\log z}+O(z^2)$$

Taking $z=(x/q)^{1/2-o(1)}$ completes the proof.

□

References

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