INTRODUCTORY LECTURES
COURSE NOTES, 2015

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1. Partial summation

Often we will evaluate sums of the form

$$\sum_{A<n\le B} a_n f(n) \qquad a_n \in \mathbb{C} \qquad f : \mathbb{Z} \to \mathbb{C}.$$

One method, which in practice is quite effective is due to Abel. We start by taking

$$S(x) = \sum_{1\le n\le x} a_n$$

and observing that

$$S(n) - S(n-1) = a_n.$$

Using this we see that for integers $B > A$

$$\begin{aligned}\sum_{A<n\le B} a_n f(n) =& \sum_{A<n\le B} f(n)(S(n) - S(n-1)) \\ =& \sum_{A<n\le B} f(n) - \sum_{A-1<n\le B-1} f(n+1)S(n) \\ =& f(B)S(B) - f(A)S(A) - \sum_{A-1<n\le B-1} S(n)(f(n+1) - f(n)).\end{aligned}$$

For an integer $n \ge 1$ and $n \le x < n+1$ one has $S(x) = S(n)$. So if f is continuously differentiable we can use the fundamental theorem of calculus to see that

$$\begin{aligned}\sum_{A-1<n\le B-1} S(n)(f(n+1) - f(n)) &= \sum_{n=A}^{B-1} S(n) \int_n^{n+1} f'(x)\,dx \\ &= \sum_{n=A}^{B-1} \int_n^{n+1} S(x)f'(x)\,dx \\ &= \int_A^B S(x)f'(x)\,dx.\end{aligned}$$

This implies the following formula for partial summation

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Theorem 1.1 (Partial summation)**.** *Suppose that* $f : \mathbb{R} \to \mathbb{C}$ *is continuously differentiable. Then*

$$\sum_{A<n\leq B} a_n f(n) = f(B)S(B) - f(A)S(A) - \int_A^B S(x)f'(x)\, dx.$$

Remark. *There is some subtlety with endpoints here. Notice that slightly altering the values of* A *and* B *may leave the left-hand side of the formula unchanged. As a consistency check verify that the value of the right-hand would also be unaltered.*

Example. Evaluate

$$\sum_{1\leq n\leq N} \log n.$$

Take $a_n = 1,\ f(n) = \log n,\ S(x) = \lfloor x \rfloor$. Here and throughout $\lfloor x \rfloor$ is the floor function and equals the largest integer $\leq x$. The partial summation formula gives

$$\begin{aligned}\sum_{1\leq n\leq N} \log n =&\lfloor N \rfloor \log N - \int_1^N \frac{\lfloor x \rfloor}{x}\, dx\\ =&N \log N - N + O(\log N).\end{aligned}$$

For a complex variable s the Riemann zeta-function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1).$$

Riemann observed that the analytic properties of $\zeta(s)$ are closely related to the distribution of the prime numbers and (amongst other things) showed that $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$. We will prove

Theorem 1.2. *The Riemann zeta-function admits an analytic continuation to the half-plane* $\mathrm{Re}(s) > 0$ *except for a simple pole at* $s = 1$*. Morever for* $\mathrm{Re}(s) > 0$ *one has*

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}}\, dx$$

where $\{x\} = x - \lfloor x \rfloor$*.*

Proof. Let s be a complex variable. Using partial summation with $S(x) = \lfloor x \rfloor$ and $f(x) = 1/x^s$ we get that

$$\sum_{1\leq n\leq N} \frac{1}{n^s} = \frac{\lfloor N \rfloor}{N^s} + s \int_1^N \frac{\lfloor x \rfloor}{x^{s+1}}\, dx.$$

Take $N \to \infty$ to get that for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\zeta(s) =& s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}\,dx\\ =& \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.\end{aligned}$$

Note that the right-hand side is analytic in the half-plane $\mathrm{Re}(s) > 0$ except for a simple pole at $s = 1$. This provides the analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$. □

2. Chebyshev's theorem and Merten's formulas

The prime number theorem states that

$$\begin{aligned}\pi(x) =& \sum_{\substack{p \le x\\ p \text{ prime}}} 1 = \mathrm{Li}(x) + O\left(x\exp(-\sqrt{\log x})\right)\\ =& \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).\end{aligned}$$

For our purposes the weaker estimate of Chebyshev will often be sufficient.

Theorem 2.1 (Chebyshev's Theorem)**.** *There exist constants $c < 1 < C$ such that*

$$\frac{cx}{\log x} \le \pi(x) \le \frac{Cx}{\log x}.$$

Remark. *Let*

$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

Using partial summation Chebyshev's estimate is equivalent to

$$c'x \le \psi(x) \le C'x.$$

We will prove Chebyshev's Theorem in this form.

Proof. Recalling $\Lambda * \mathbf{1} = \log$ we have

$$\begin{aligned}\sum_{n\le x} \log n =& \sum_{n \le x}\sum_{ab=n} \Lambda(a) = \sum_{b \le x}\sum_{a \le x/b} \Lambda(a)\\ =& \sum_{b \le x} \psi\left(\frac{x}{b}\right) = \sum_{b=1}^{\infty} \psi\left(\frac{x}{b}\right).\end{aligned}$$

Therefore

$$\sum_{b=1}^{\infty} \psi\left(\frac{x}{b}\right) = x\log x - x + O(\log x). \tag{1}$$

Apply (1) twice to get that

(2)
$$\begin{aligned}\sum_{b=1}^{\infty}\psi\left(\frac{2N}{b}\right)-2\sum_{b=1}^{\infty}\psi\left(\frac{N}{b}\right)=&2N\log 2N-2N-2(N\log N-N)+O(\log N)\\=&N\log 4+O(\log X).\end{aligned}$$

Combining the even terms from the first sum with the second sum gives

$$\sum_{b=1}^{\infty}\left(\psi\left(\frac{2N}{2b-1}\right)-\psi\left(\frac{N}{b}\right)\right)=N\log 4+O(\log N). \tag{3}$$

The function $\psi(x)$ is non-decreasing so each term in the above sum is positive. Thus dropping all but the first term

$$\psi\left(2N\right)-\psi\left(N\right)\leq N\log 4+O(\log N). \tag{4}$$

Using this relation at $N=x/2,x/4,x/8,\dots,x/2^A$ where $A=\lfloor\log x/\log 2\rfloor$ and summing gives

$$\sum_{b=1}^{A}\left(\psi\left(\frac{x}{2^b}\right)-\psi\left(\frac{x}{2^{b-1}}\right)\right)\leq x\log 4\sum_{b=1}^{\infty}\frac{1}{2^b}+O((\log x)^2).$$

Therefore

$$\psi\left(x\right)\leq x\log 4+O((\log x)^2). \tag{5}$$

Next rewrite (3) to see

$$\psi(2N)-\sum_{b=1}^{\infty}\left(\psi\left(\frac{N}{b}\right)-\psi\left(\frac{2N}{2b+1}\right)\right)=N\log 4+O(\log N).$$

Every term in the sum on the right hand side is positive so that applying this at $N=x/2$

$$\psi(x)\geq x\log 2+O(\log N).$$

$\square$

From the proof it follows that by (3) and (5)

$$\begin{aligned}\psi(2x)-\psi(x)\geq&x\log 4-\psi(2x/3)+O(\log x)\\\geq&\left(\frac{1}{3}\log 4\right)x+O(\log x)\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{x<p\leq 2x}1\geq&\frac{1}{\log 2x}\sum_{x<p\leq 2x}\log p\\\geq&\frac{1}{\log 2x}\left(\psi(2x)-\psi(x)+O(\sqrt{x}\log x\right)\\\geq&\left(\frac{1}{3}\log 4\right)\frac{x}{\log x}(1+o(1)).\end{aligned}$$

Corollary 2.2 (Bertrand's postulate)**.** *For each real number $x \geq 1$ there is a prime number in the interval $[x, 2x]$.*

Remark. *Bertrand's postulate has been significantly improved. For any sufficiently large x it is known that there exists $\theta < 1$ such that there is a prime number in every interval of the form $[x, x+x^\theta]$. The best known result in this direction gives $\theta = 21/40$ and it is conjectured that this should hold for any $\theta > 0$.*

Using the prime number theorem and partial summation it is straightforward to check that

$$\sum_{p \leq x} \frac{1}{p} = \int_2^x \frac{dt}{t \log t}(1 + o(1)) = \log\log x(1 + o(1)).$$

However, in this instance Chebyshev's theorem suffices to establish

Theorem 2.3 (Mertens' formulas)**.** *We have*

a) $\displaystyle\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1)$.

b) $\displaystyle\sum_{p \leq x} \log p = \log x + O(1)$.

c) $\displaystyle\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \asymp \frac{1}{\log x}$.

Remarks. For $f, g > 0$ the notation $f(x) \asymp g(x)$ means there exist constants c_1, c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for all x under consideration.

From part *c*) it immediately follows that $\phi(n) \gg n/\log\log n$, $(n \geq 3)$. To see this note that since the number of prime divisors of n is $\leq C \log n$ (for some $C > 1$) we have

$$\frac{\phi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right) \geq \prod_{p \leq C \log n} \left(1 - \frac{1}{p}\right) \gg \frac{1}{\log\log n}.$$

Additionally, it is possible to give more precise formulas than those given above. In particular, it is known that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + b + O(1/\log x)$$

where b is a certain absolute constant. Also,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x},$$

where γ is Euler's constant.

Proof. We first will establish *b*). The argument is similar to the one given to prove Chebyshev's theorem. Use the relation $\log = \Lambda * \mathbf{1}$ and switch order of summation

$$\begin{aligned}\frac{1}{x}\sum_{n\le x}\log n =&\frac{1}{x}\sum_{n\le x}\sum_{ab=n}\Lambda(a)\\ =&\frac{1}{x}\sum_{a\le x}\Lambda(a)\sum_{b\le x/a}1\\ =&\sum_{a\le x}\frac{\Lambda(a)}{a}+O\left(\frac{\psi(x)}{x}\right).\end{aligned}$$

Evaluate the left-hand side using partial summation and apply Chebyshev's theorem to get

$$\sum_{a\le x}\frac{\Lambda(a)}{a}=\log x+O(1).$$

Observe that

$$\sum_{a\le x}\frac{\Lambda(a)}{a}=\sum_{p\le x}\frac{\log p}{p}+\sum_{\substack{p^n\le x\\ n\ge 2}}\frac{\log p}{p}.$$

The second sum is clearly $O(1)$. This gives b).

Once again bounding the higher prime powers we see

$$\sum_{p\le x}\frac{1}{p}=\sum_{n\le x}\frac{\Lambda(n)}{n\log n}+O(1).$$

Now use partial summation with $a_n=\Lambda(n)/n$, and $f(x)=1/(\log x)$ to get

$$\begin{aligned}\sum_{n\le x}\frac{\Lambda(n)}{n\log n}=&\frac{1}{\log x}(\log x+O(1))+\int_2^x\frac{(\log t+O(1))}{t(\log t)^2}\,dt\\ =&\log\log x+O(1).\end{aligned}$$

To establish part c) we note that

$$\begin{aligned}\prod_{p\le x}\left(1-\frac{1}{p}\right)=&\exp\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)\right)\\ =&\exp\left(-\sum_{p\le x}\frac{1}{p}+O(1)\right)\\ =&\exp\left(-\log\log x+O(1)\right)\asymp\frac{1}{\log x}.\end{aligned}$$

$\square$