NORMAL ORDER OF $\omega(n)$ AND THE ERDŐS-KAC THEOREM

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1. Normal order of $\omega(n) = \sum_{p|n} 1$

How many prime divisors does a typical integer have? Applying Mertens' theorem

$$\begin{aligned}\sum_{n\le x}\omega(n) =& \sum_{n\le x}\sum_{p|n} 1\\ =&\sum_{p\le x}\left(\frac{x}{p}+O(1)\right)\\ =&x\log\log x + O(x).\end{aligned}$$

This shows that the average number of prime divisors of n for $n \le x$ is $\log\log x$.

In this section we will see that for almost all n that $\omega(n) = \log\log n(1 + o(1))$. This means very large integers typically have very few prime factors.

Definition. *The **density** (sometimes called arithmetic density) of a subset $\mathcal{S} \subset \mathbb{N}_{\ge 1}$ is*

$$\lim_{N\to\infty}\frac{\mathcal{S}\bigcap\{1,2,\ldots,N\}}{N}$$

provided that the limit exists.

Examples.

- the density of the even positive integers is $1/2$
- earlier we saw that the density of the square-free numbers is $1/\zeta(2)$.
- Chebyshev's theorem implies the density of the prime numbers is zero.

In this section we will a give a proof of the following theorem of Hardy and Ramanujan [8].

Theorem 1.1 (Hardy-Ramanujan)**.** *There exists a density one subsequence of positive integers $\{n_j\}$ such that*

$$\omega(n_j) = \log\log n_j(1 + o(1)) \qquad (j\to\infty).$$

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To prove the theorem we will calculate the variance of $\omega(n)$ and apply Chebyshev's inequality. This proof was discovered some years later by Turán [11]. A nice account of the history of this theorem is given in Elliott's book [4].

Lemma 1.2 (Turán)**.**

$$\sum_{n\le x}(\omega(n)-\log\log x)^2 = O(x\log\log x) \tag{1}$$

Proof. Squaring out gives

$$\sum_{n\le x}(\omega(n)-\log\log x)^2 = \sum_{n\le x}\omega(n)^2 - 2\log\log x\sum_{n\le x}\omega(n) + \lfloor x\rfloor(\log\log x)^2.$$

By the lemma we can evaluate the middle term so the left-hand side equals

$$\sum_{n\le x}\omega(n)^2 - x(\log\log x)^2 + O(x\log\log x).$$

To establish (1) it suffices to show that

$$\sum_{n\le x}\omega(n)^2 \le x(\log\log x)^2 + O(x\log\log x). \tag{2}$$

Observe that

$$\begin{aligned}\sum_{n\le x}\omega(n)^2 &= \sum_{n\le x}\left(\sum_{p_1|n}1\right)\left(\sum_{p_2|n}1\right)\\ &= \sum_{p_1\le x}\sum_{p_2\le x}\sum_{\substack{n\le x\\ p_1|n,p_2|n}}1.\end{aligned} \tag{3}$$

First, we estimate the terms with $p_1=p_2$

$$\sum_{p\le x}\sum_{\substack{n\le x\\ p|n}}1 = \sum_{p\le x}\left(\frac{x}{p}+O(1)\right) = x\log\log x + O(x). \tag{4}$$

Next to estimate the terms with $p_1 \neq p_2$ note that both p_1 and p_2 divide n iff $p_1 p_2 | n$. This gives that

$$\begin{aligned}
\sum_{\substack{p_1,p_2\le x\\ p_1\neq p_2}} \sum_{\substack{n\le x\\ p_1|n,\, p_2|n}} 1 &= \sum_{\substack{p_1p_2\le x\\ p_1\neq p_2}} \sum_{\substack{n\le x\\ p_1p_2|n}} 1\\
&= \sum_{\substack{p_1p_2\le x\\ p_1\neq p_2}} \left(\frac{x}{p_1p_2} - \left\{\frac{x}{p_1p_2}\right\}\right)\\
&\le x \sum_{p_1p_2\le x} \frac{1}{p_1p_2}\\
&\le x \left(\sum_{p\le x}\frac{1}{p}\right)^2\\
&\le x(\log\log x)^2 + O(x\log\log x).
\end{aligned} \tag{5}$$

Applying (4) and (5) in (3) implies (2). □

Proof of the Hardy-Ramanujan theorem. Let

$$\mathcal{S} = \left\{ n \ge 3 : |\omega(n) - \log\log n| \ge (\log\log n)^{3/4} \right\}$$

Note that if $n \notin \mathcal{S}$, $(n \ge 3)$, then $\omega(n) = \log\log n(1 + O(1/(\log\log n)^{1/4})$. It suffices to prove that the density of S equals zero. Apply Chebyshev's inequality to get

$$\begin{aligned}
\sum_{\substack{1\le n\le N\\ n\in\mathcal{S}}} 1 &= \sum_{\substack{\sqrt{N}<n\le N\\ |\omega(n)-\log\log n|\ge(\log\log n)^{3/4}}} 1 + O\left(\sqrt{N}\right)\\
&\le \sum_{\substack{\sqrt{N}<n\le N\\ |\omega(n)-\log\log n|\ge(\log\log n)^{3/4}}} \left|\frac{\omega(n)-\log\log n}{(\log\log n)^{3/4}}\right|^2 + O\left(\sqrt{N}\right).
\end{aligned}$$

For $\sqrt{N} < n \le N$ that $\log\log n = \log\log N + O(1)$. Using this and applying Lemma 1.2 yields

$$\begin{aligned}
\sum_{\substack{1\le n\le N\\ n\in\mathcal{S}}} 1 &\ll \frac{1}{(\log\log N)^{3/2}} \sum_{\sqrt{N}<n\le N} (\omega(n) - \log\log N + O(1))^2 + \sqrt{N}\\
&\ll \frac{N}{(\log\log N)^{1/2}} = o(N).
\end{aligned}$$

The implies the density of $\mathcal{S}$ is zero, which establishes the claim. □

2. The Erdős-Kac central limit theorem

In the last section we estimated the mean and variance of $\omega(n)$. The higher moments have been estimated as well (by Halberstam [7]). It has been proved that for each integer $k \geq 1$, as $x \to \infty$

$$\frac{1}{x}\sum_{n\leq x}(\omega(n)-\log\log x)^k = m_k(\log\log x)^{k/2}+o((\log\log x)^{k/2}), \tag{6}$$

where m_k are the moments of the standard normal random variable (mean zero variance one)

$$m_k = \begin{cases} \frac{(2\ell)!}{2^\ell \ell!} \text{ if } k=2\ell, \\ 0 \text{ otherwise.}\end{cases}$$

Hence, the moments of

$$\frac{\omega(n)-\log\log x}{\sqrt{\log\log x}}$$

are asymptotically equal to the moments of the normal random variable. Since the normal distribution is determined by its moments (this is not always the case!) we have

Theorem 2.1 (Erdős-Kac central limit theorem). *As* $x \to \infty$

$$\frac{1}{x}\#\left\{1 \leq n \leq x : \frac{\omega(n)-\log\log x}{\sqrt{\log\log x}} \leq a\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-t^2/2}\,dt(1+o(1)).$$

Remarks. The original proof of the given by Erdős and Kac [5] follows from different techniques. Other proofs of this theorem have also been given by Halberstam [7], Delange [2] (as well as Selberg), Khan [10], Harper [9], and Granville and Soundararajan [6]. Notably, Delange [2] gives an asymptotic expansion for the distribution function of $\frac{\omega(n)-\log\log x}{\sqrt{\log\log x}}$. The proof that we shall give follows (more or less) the argument of Granville and Soundararajan [6].

2.1. **A probabilistic heuristic.** Let

$$g_p(n) = \begin{cases} 1 \text{ if } p|n \\ 0 \text{ otherwise.}\end{cases}$$

Intuitively we expect the following:

- the probability that an integer is divisible by p is $1/p$
- for $p \neq q$ the event that $p|n$ is independent from the event $q|n$

This suggests the following probabilistic model for $g_p(n)$. For primes p let $X(p)$ be independent Bernoulli random variables such that

$$X(p) = \begin{cases} 1 \text{ with probability } \dfrac{1}{p} \\ 0 \text{ with probability } 1-\dfrac{1}{p}.\end{cases}$$

The expectation of $X(p)$ is

$$\mathbb{E}(X(p)) = 1 \cdot \frac{1}{p} + 0 \cdot \left(1 - \frac{1}{p}\right) = \frac{1}{p}$$

and the variance is

$$\begin{aligned}\sigma_p^2 = \mathbb{E}\left(\left(X(p) - \frac{1}{p}\right)^2\right) =& \mathbb{E}\left(X(p)^2\right) - \frac{2}{p}\mathbb{E}\left(X(p)\right) + \frac{1}{p^2} \\ =& \frac{1}{p}\left(1 - \frac{1}{p}\right).\end{aligned}$$

Also,

$$\sum_{p\le P} \sigma_p^2 = \sum_{p\le P} \frac{1}{p}\left(1 - \frac{1}{p}\right) = \log\log P + O(1). \tag{7}$$

The Lindeberg-Feller central limit theorem (see Theorem 27.2 of [1] or Theorem 3.4.5 of [3]) from probability theory implies that

$$\mathbb{P}\left(\frac{\sum_{p\le P}\left(X_p - \frac{1}{p}\right)}{\sqrt{\log\log P}} \le a\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-t^2/2}\, dt(1 + o(1)) \qquad (P \to \infty).$$

Additionally, since $|X_p| \le 1$ it also follows that (see the second proof of the Lindeberg-Feller theorem given in [3]) as $P \to \infty$

$$\mathbb{E}\left(\left(\sum_{p\le P}\left(X_p - \frac{1}{p}\right)\right)^k\right) = \begin{cases} \frac{(2\ell)!}{2^\ell \ell!}(\log\log P)^\ell(1 + o(1)) \text{ if } k = 2\ell, \\ o((\log\log P)^{k/2}) \text{ otherwise.}\end{cases} \tag{8}$$

These formulas are also not difficult to derive by direct methods, which we will do in the appendix below.

To see how this relates to the Erdős-Kac CLT notice that

$$\omega(n) - \log\log x = \sum_{p\le x}\left(g_p(n) - \frac{1}{p}\right) + O(1).$$

We will prove below that

$$\frac{1}{x}\sum_{n\le x}\left(\sum_{p\le z}\left(g_p(n) - \frac{1}{p}\right)\right)^k = \mathbb{E}\left(\left(\sum_{p\le z}\left(X(p) - \frac{1}{p}\right)\right)^k\right) + O\left(C^k \frac{z^k}{x(\log z)^k}\right).$$

From this formula we will then establish (6).

2.2. **Proof of Erdős-Kac CLT.** To shorten notation write $f_p(n) = g_p(n) - 1/p$.

Lemma 2.2. *Let* $z = x^{1/\log\log\log x}$ *then for* $n \le x$

$$\omega(n) - \log\log x = \sum_{p\le z} f_p(n) + O\left(\log\log\log x\right)$$

Proof. Write

$$\omega(n) - \log\log x = \sum_{\substack{p|n \\ p\le z}} 1 + \sum_{\substack{p|n \\ p>z}} 1 - \log\log x$$

Combining the first sum with $-\log\log x$ gives

$$\omega(n) - \log\log x = \sum_{p\le z} f_p(n) + \sum_{\substack{p|n \\ p>z}} 1 + O(1)$$

To complete the proof note that since $n \le x$ the number of distinct prime factors of n with $p > z$ must be $O(\log\log\log x)$ (since if not then we could multiply all these primes together to get a number bigger than x, but we are assuming $n \le x$). □

Lemma 2.3. *Let $1 \le a \le x$ then*

$$\sum_{\substack{n\le x \\ (n,a)=1}} 1 = x\frac{\phi(a)}{a} + O(\tau(a)).$$

Proof. Use the the mobius inversion formula $\mathbf{1} * \mu = \delta$ to get that

$$\begin{aligned}
\sum_{\substack{n\le x \\ (n,a)=1}} 1 =& \sum_{n\le x} \delta((n,a)) \\
=& \sum_{n\le x} \sum_{d|(n,a)} \mu(d) \\
=& \sum_{\substack{d\le x \\ d|a}} \mu(d) \sum_{\substack{n\le x \\ d|n}} 1 \\
=& x\sum_{d|a} \frac{\mu(d)}{d} + O(\tau(a)).
\end{aligned}$$

Now use the exercise from the homework to complete the proof. □

Write $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ We now extend the definition of $f_p(n)$ by setting

$$f_m(n) := \prod_{j=1}^{r} (f_{p_j}(n))^{\alpha_j}$$

Proposition 2.4. *Suppose that $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $M = p_1 \cdots p_r$. Then*

$$\frac{1}{x}\sum_{n\le x} f_m(n) = \prod_{j=1}^{r} \left(\frac{1}{p_j}\left(1 - \frac{1}{p_j}\right)^{\alpha_j} + \left(\frac{-1}{p_j}\right)^{\alpha_j} \left(1 - \frac{1}{p_j}\right)\right) + O\left(\frac{2^{2r}}{x}\right)$$

Proof. If $(n, M) = d$ then $f_m(n) = f_m(d)$ so that applying the previous lemma we get

$$\begin{aligned}\sum_{n\le x} f_m(n) =& \sum_{d|M} \sum_{\substack{n\le x\\(M,n)=d}} f_m(n)\\ =& \sum_{d|M} f_m(d) \sum_{\substack{n/d\le x/d\\(M/d,n/d)=1}} 1\\ =& x\sum_{d|M} f_m(d)\frac{\phi(M/d)}{M} + O(\tau(M)^2),\end{aligned}$$

where we used the fact that $|f_m(d)| \le 1$ to bound the error term and note that $\tau(M) = 2^r$. Next use the definition of $f_m(n)$ to see that

$$\begin{aligned}\sum_{d|M} f_m(d)\frac{\phi(M/d)}{M} =& \sum_{d|M}\prod_{p_j|d}\left(1-\frac{1}{p_j}\right)^{\alpha_j}\prod_{p_j|R/d}\left(\frac{-1}{p_j}\right)^{\alpha_j}\frac{\phi(M/d)}{M}\\ =& \sum_{d|M}\prod_{p_j|d}\frac{1}{p_j}\left(1-\frac{1}{p_j}\right)^{\alpha_j}\prod_{p_j|R/d}\left(\frac{-1}{p_j}\right)^{\alpha_j}\left(1-\frac{1}{p_j}\right).\end{aligned}$$

Since M is a product of distinct primes $d = \prod_{j\in S} p_j$ for some $S \subset \{1, 2, \ldots, r\}$ so we have

$$\begin{aligned}&\sum_{S\subset\{1,2,\ldots,r\}}\prod_{j\in S}\frac{1}{p_j}\left(1-\frac{1}{p_j}\right)^{\alpha_j}\prod_{j\in S^c}\left(\frac{-1}{p_j}\right)^{\alpha_j}\left(1-\frac{1}{p_j}\right)\\ &= \prod_{j=1}^{r}\left(\frac{1}{p_j}\left(1-\frac{1}{p_j}\right)^{\alpha_j} + \left(\frac{-1}{p_j}\right)^{\alpha_j}\left(1-\frac{1}{p_j}\right)\right).\end{aligned}$$

□

Note that

$$\begin{aligned}\mathbb{E}\left(\prod_{j=1}^{r}\left(X(p_j)-\frac{1}{p_j}\right)^{\alpha_j}\right) =& \prod_{j=1}^{r}\mathbb{E}\left(\left(X(p_j)-\frac{1}{p_j}\right)^{\alpha_j}\right)\\ =& \prod_{j=1}^{r}\left(\frac{1}{p_j}\left(1-\frac{1}{p_j}\right)^{\alpha_j} + \left(\frac{-1}{p_j}\right)^{\alpha_j}\left(1-\frac{1}{p_j}\right)\right).\end{aligned}$$

So for $m = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$

$$\frac{1}{x}\sum_{n\le x} f_m(n) = \mathbb{E}\left(\prod_{j=1}^{r}\left(X(p_j)-\frac{1}{p_j}\right)^{\alpha_j}\right) + O\left(\frac{2^{2r}}{x}\right). \tag{9}$$

Corollary 2.5. *For integers $k \geq 1$ and $z = x^{1/\log\log\log x}$*

$$\frac{1}{x}\sum_{n\leq x}\left(\sum_{p\leq z} f_p(n)\right)^k = \mathbb{E}\left(\sum_{p\leq z}\left(X(p)-\frac{1}{p}\right)\right)^k + O\left(C^k \frac{z^k}{x(\log z)^k}\right)$$
$$= \begin{cases} \frac{(2\ell)!}{2^\ell \ell!}(\log\log x)^\ell(1+o(1)) & \text{if } k = 2\ell, \\ o((\log\log x)^{k/2}) & \text{otherwise.}\end{cases}$$

Proof. We have

$$\frac{1}{x}\sum_{n\leq x}\left(\sum_{p\leq z} f_p(n)\right)^k = \sum_{p_1,\ldots,p_k\leq z}\frac{1}{x}\sum_{n\leq x} f_{p_1\cdots p_k}(n)$$
$$= \sum_{p_1,\ldots,p_k\leq z}\left(\mathbb{E}\left(\prod_{j=1}^r\left(X(p_j)-\frac{1}{p_j}\right)^{\alpha_j}\right) + O\left(\frac{2^{2k}}{x}\right)\right)$$
$$= \mathbb{E}\left(\sum_{p\leq z}\left(X_p - \frac{1}{p}\right)\right)^k + O\left(C^k\frac{z^k}{x(\log z)^k}\right)$$

We now apply (8) with $P = z$. □

Proof of the Erdős-Kac CLT. By the earlier lemma

$$\frac{1}{x}\sum_{n\leq x}(\omega(n)-\log\log x)^k = \frac{1}{x}\sum_{n\leq x}\left(\sum_{p\leq z} f_p(n) + R(n,x)\right)^k$$
$$= \frac{1}{x}\sum_{n\leq x}\sum_{j=0}^k\binom{k}{j}\left(\sum_{p\leq z} f_p(n)\right)^j R(n,x)^{k-j}$$
$$= \frac{1}{x}\sum_{n\leq x}\left(\sum_{p\leq z} f_p(n)\right)^k + \mathcal{E}$$

where

$$\mathcal{E} \ll 2^k(\log\log\log x)^k \max_{j=1,\ldots,k-1}\frac{1}{x}\sum_{n\leq x}\left|\sum_{p\leq z} f_p(n)\right|^k$$

By Cauchy-Schwarz

$$\frac{1}{x}\sum_{n\leq x}\left|\sum_{p\leq z} f_p(n)\right|^j \ll \sqrt{\frac{1}{x}\sum_{n\leq x}\left|\sum_{p\leq z} f_p(n)\right|^{2j}} \ll \left(\frac{(2j)!}{2^j j!}\right)^{1/2}(\log\log x)^{j/2}$$

So that

$$\mathcal{E} \ll_k (\log\log\log x)^k(\log\log x)^{(k-1)/2}.$$

Applying Corollary 2.5 completes the proof.

□

2.3. **Appendix: Direct calculations of moments.** In this section we will show how to directly estimate the moments

$$\mathbb{E}\left(\left(\sum_{p\le z}\left(X(p)-\frac{1}{p}\right)\right)^k\right).$$

We will do this via the moment generating function

$$\mathbb{E}\left(\exp\left(\zeta\cdot\left(\frac{\sum_{p\le z}\left(X(p)-\frac{1}{p}\right)}{\sqrt{\log\log z}}\right)\right)\right)$$

where ζ is a complex variable. We first observe that it suffices to prove that uniformly on compact subsets of $\mathbb{C}$

$$\mathbb{E}\left(\exp\left(\zeta\cdot\left(\frac{\sum_{p\le z}\left(X(p)-\frac{1}{p}\right)}{\sqrt{\log\log z}}\right)\right)\right)\to e^{\zeta^2/2}\qquad (z\to\infty).$$

To see this differentiate to get that

$$\mathbb{E}\left(\left(\frac{\sum_{p\le z}\left(X(p)-\frac{1}{p}\right)}{\sqrt{\log\log z}}\right)^k\right)=\frac{d^k}{d\zeta^k}\mathbb{E}\left(\exp\left(\zeta\frac{\sum_{p\le z}\left(X(p)-\frac{1}{p}\right)}{\sqrt{\log\log z}}\right)\right)\Big|_0\to\frac{d^k}{d\zeta^k}e^{\zeta^2/2}\Big|_0.$$

Now apply Cauchy's formula to see that

$$\frac{d^k}{d\zeta^k}e^{\zeta^2/2}\Big|_0=\frac{k!}{2\pi i}\int_{|z|=1}e^{z^2/2}\frac{dz}{z^{k+1}}=k!\sum_{n=0}^{\infty}\frac{1}{2^n n!}\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{z^{k-2n+1}}=m_k.$$

We now estimate the moment generating function. By independence

$$\mathbb{E}\left(\exp\left(\zeta\frac{\sum_{p\le P}X_p-\frac{1}{p}}{\sqrt{\log\log P}}\right)\right)=\prod_{p\le P}\mathbb{E}\left(\exp\left(\zeta\frac{X_p-\frac{1}{p}}{\sqrt{\log\log P}}\right)\right).$$

Now assume that $|\zeta|\le(\log\log P)^{1/4}$ (and note $|X_p|\le 1$) so that

$$\begin{aligned}\mathbb{E}\left(\exp\left(\zeta\frac{X_p-\frac{1}{p}}{\sqrt{\log\log P}}\right)\right)=&\mathbb{E}\left(1+\zeta\frac{(X_p-\frac{1}{p})}{\sqrt{\log\log P}}+\zeta^2\frac{(X_p-\frac{1}{p})^2}{2\log\log P}+O\left(\left(\frac{|\zeta|^3}{(\log\log P)^{3/2}}\right)\right)\right)\\=&1+\frac{\zeta^2}{2}\cdot\frac{\frac{1}{p}(1-\frac{1}{p})}{\log\log P}+O\left(\frac{|\zeta|^3}{(\log\log P)^{3/2}}\right)\end{aligned}$$

where we have used (7). Thus, for $|\zeta| \leq (\log\log P)^{1/4}$

$$\begin{aligned}\mathbb{E}\left(\exp\left(\zeta\frac{\sum_{p\leq P} X_p - \frac{1}{p}}{\sqrt{\log\log P}}\right)\right) &= \prod_{p\leq P}\left(1+\frac{\zeta^2}{2}\cdot\frac{\frac{1}{p}(1-\frac{1}{p})}{\log\log P}+O\left(\frac{|\zeta|^3}{(\log\log P)^{3/2}}\right)\right)\\ &=\exp\left(\frac{\zeta^2}{2\log\log P}\sum_{p\leq P}\frac{1}{p}(1-\frac{1}{p})+O\left(\frac{|\zeta|^3}{(\log\log P)^{3/2}}\right)\right),\\ &=\exp\left(\frac{\zeta^2}{2}+O\left(\frac{1}{(\log\log P)^{1/2}}\right)\right)\\ &=e^{\zeta^2/2}\left(1+O\left(\frac{1}{(\log\log P)^{1/2}}\right)\right).\end{aligned}$$

which establishes the claim.

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