Instructions: The assignment should be delivered to my mailbox or sent via email (rudnick@post.tau.ac.il) by Wednesday, 15 July 2015 at latest.

Exercise 1. For $N \geq 1$, the Farey sequence of level $N$ is defined to be all rationals in $(0, 1]$ which in reduced form have denominator at most $N$:

$$\mathcal{F}_N = \left\{ \frac{a}{q} : \gcd(a,q) = 1, \ 1 \leq a \leq q \leq N \right\}.$$

For instance, $\mathcal{F}_5 = \left\{ 1_5, 1_4, 1_3, 2_5, 1_2, 3_5, 2_3, 3_4, 4_5, 1_1 \right\}$. Show that

$$\# \mathcal{F}_N = \frac{N^2}{2\zeta(2)} + O(N \log N).$$

Hint: $\varphi(n) = n \sum_{d|n} \mu(d)/d$.

Exercise 2. Show that

$$\sum_{n \leq x} \Lambda(n)^2 = x \log x - x + o(x)$$

where $\Lambda(n)$ is the von Mangoldt function.

Exercise 3. A Carmichael number is a composite integer $N$ for which $a^{N-1} \equiv 1 \mod N$ for all $a$ coprime to $N$.

a) Show that if $N$ is a Carmichael number, then $N$ is odd, and that for all prime divisors $p | N$, we have $p - 1 \mid N - 1$ (this is a converse of Korselt’s criterion).

b) Show that for $p > 3$, if $p, q := 2p - 1$ and $r := 3p - 2$ are all prime, then their product $N = pqr$ is a Carmichael number.

Exercise 4. Show that the number of integers $n \leq x$ so that $n$, $2n - 1$, $3n - 2$ are all primes, is at most $\ll x/(\log x)^3$.

Exercise 5. For a monic integer polynomial

$$f(t) = t^n + a_{n-1}t + \cdots + a_0$$

we define the height as $\text{Ht}(f) = \max_j |a_j|$. We define

$$\mathcal{R}_n(N) = \{ f(t) = t^n + a_{n-1}t + \cdots + a_0 : \text{Ht}(f) \leq N; \text{reducible over } \mathbb{Q} \}$$

to be the set of reducible monic polynomial of degree $n$ with integer coefficients, of height at most $N$. Show that for $n > 2$,

$$\# \mathcal{R}_n(N) \ll_n N^{n-\frac{1}{2}} \log N.$$

Hint: Use the large sieve in its $n$-dimensional form, where

$$\Omega_p = \{ f \in \mathbb{F}_p[t], \deg f = n, \text{monic irreducible over } \mathbb{F}_p \}.$$