Exercise 1. Let $\omega(n)$ the number of distinct prime divisors of an integer $n$, so for instance $\omega(12) = 2$. Show that $\omega(n) \ll \log n / \log \log n$.

Exercise 2. Let $\Omega(n)$ be the number of prime divisors of $n$, counted with multiplicity, i.e. the number of prime powers dividing $n$, so for instance $\Omega(12) = 3$. Show that the mean value of $\Omega(n)$ is $\log \log n$:

$$\frac{1}{N} \sum_{n \leq N} \Omega(n) = \log \log N + O(1).$$

Exercise 3. Let $\tau(n) = \sum_{d|n} 1$ be the divisor function. Using $\sum_{n \leq x} \tau(n) = x(\log x + C) + O(x^{1/2})$, show that

$$\sum_{n > Y} \tau(n) n^{-2} = \frac{\log Y + C + 2}{Y} + O\left( \frac{1}{Y^{3/2}} \right).$$

Exercise 4. Let $a > 0$, $b$ be coprime integers. Find the density of integers $n$ for which $an + b$ is squarefree.

Exercise 5. Let $f(x) \in \mathbb{Z}[x]$ be a separable polynomial (i.e. with no repeated roots) of positive degree. Set $B := \gcd\{f(n) : n \in \mathbb{Z}\}$ and let $B'$ be the smallest divisor of $B$ so that $B/B'$ is square-free. For each prime $p$, we denote by $p^{\nu_p}$ the largest power of $p$ dividing $B'$, and by $r_f(p)$ the number of $a \mod p^{\nu_p}$ for which $f(a)/B'$ is squarefree. We set

$$c_f = \prod_p \left( 1 - \frac{r_f(p)}{p^{\nu_p}} \right)$$

which is the conjectural density of integers $n$ for which $f(n)/B'$ is squarefree.

For $f(x) = x(x+1)(x+2)(x+3)$, find $B_f$ and $B'_f$, and show that $r_f(p) = 4$ for $p \neq 2, 3$.

Hence $c_f = R \prod_{p \neq 2,3} (1 - \frac{4}{p^2})$ for some rational number. Find $R$.

Exercise 6. Assume that $B' = 1$. Show that $c_f > 0$, i.e. that $r_f(p) < p^2$ for all primes $p$. 