SIEVE THEORY 2015
ASSIGNMENT 3
DUE DATE: WEDNESDAY, MAY 27, 2015

Exercise 1. For $m, n \geq 1$, let $[m, n] = \mathrm{lcm}(m, n)$ be their least common multiple, that is the smallest integer divisible by both m and n in the sense that it divides any other integer with this property.

a) Show that $[m, n] \cdot (m, n) = mn$, where $(m, n) = \gcd(m, n)$.

b) If f is a multiplicative function, show that for all $m, n \geq 1$,

$$f([m, n])f((m, n)) = f(m)f(n)$$

c) Show that $[m^2, n^2] = [m, n]^2$.

d) If D is squarefree, show that $\#\{((m, n) : [m, n] = D\} = 3^{\omega(D)}$ where $\omega(D)$ is the number of distinct prime divisors of D.

Exercise 2. Let $f(n) = n^2$.

a) Show that

$$(f * \mu)(n) = n^2 \prod_{p|n}(1 - \frac{1}{p^2})$$

b) Show that

$$\sum_{n \leq z} \frac{\mu(n)^2}{(\mu * f)(n)} \geq \zeta(2) + O(\frac{1}{z})$$

Exercise 3. Let $\tau(n)$ be the number of divisors of n. Show that

$$\sum_{\substack{n \leq x \\ \gcd(3,n)=1}} \frac{\tau(n)}{n} = \frac{2}{9}(\log x)^2 + O(\log x)$$