Review of the basic theory of continued fractions: For an irrational real number \( x \in \mathbb{R} \setminus \mathbb{Q} \), the continued fraction expansion is defined by

\[
x = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}
\]

where \( a_j \) are integers (the “partial quotients”), \( a_1, a_2, a_3, \ldots \geq 1 \), defined by writing

\[
x = a_0 + \frac{1}{x_1}, \quad x_1 = a_1 + \frac{1}{x_2}, \ldots
\]

with \( x_n > 1 \), so that for \( n \geq 1 \), \( a_n = [x_n] \), \( x_{n+1} = 1/\{x_n\} \). In particular

\[
x = [a_0; a_1, a_2, \ldots, a_{n-1}, x_n].
\]

We also define recursively \( p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1 \), and for \( n \geq 0 \)

\[
p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}.
\]

**Exercise 1.**

a) Show that \( p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1} \).

b) Show that

\[
[a_0; a_1, a_2, a_3, \ldots, a_n] = \frac{p_n}{q_n} \quad \text{and} \quad x = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}.
\]

**Exercise 2.**

a) Show

\[
x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(x_{n+1}q_n + q_{n-1})}
\]

b) Show that

\[
|x - \frac{p_n}{q_n}| \leq \frac{1}{a_{n+1}q_n^2}.
\]

Hence the “partial convergents” \( p_n/q_n \) give good rational approximations to \( x \): \( |x - p_n/q_n| < 1/q_n^2 \).

**Exercise 3.** Let \( x \) be an algebraic number of degree \( d \geq 2 \), with partial convergents \( p_n/q_n \). Show that \( \log q_n \ll d^n \).

Hint: Use Liouville’s theorem and Exercise 2.